The Construction of Nonspherical Models of Quasi-relaxed Stellar Systems

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Outline

- Spherical models of **collisionless** but **quasi-relaxed stellar systems** have long been studied as a natural framework for the description of globular clusters. Here we consider the construction of self-consistent models under the same physical conditions, but including the ingredients that lead to departures from spherical symmetry.
- Focus on the effects of the **tidal field** associated with a circular orbit of the cluster inside the hosting galaxy.
- The method developed here can also be used to construct models for which the nonspherical shape is due to **internal (rigid) rotation**.
- We hope these models will be a useful tool to investigate the primary cause of the flattening of globular clusters, still to be established [1].
- **Details can be found in Bertin, G. & Varri, A. L. 2008, ApJ, 689, 1005.**

1. The physical model

We consider a globular cluster on a circular orbit of radius R_0 inside a spherical galaxy, taken to be represented as a "frozen" external potential Φ_G with associated circular orbital frequency Ω. In the so-called "tidal approximation", the relevant **Jacobi integral**, with respect to a frame of reference centered on the center of mass of the globular cluster, is given by:

$$H = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2 + \Phi_T + \Phi_C \, , \tag{1}$$

$$\Phi_T = \Omega^2 \left(z^2 - \nu x^2 \right) /2 \tag{2}$$

with $\nu = 4 - \kappa^2/\Omega^2$, where κ is the epicyclic frequency at R_0. Note that we assume that $\tau_{dyn} \ll 2\pi/\Omega$.

Distribution function

We focus on the extension of spherical King models [2]; therefore, the relevant distribution function is given by:

$$f_K(H) = A[\exp(-aH) - \exp(-aH_0)] \tag{3}$$

if $H \leq H_0$ and $f_K(H) = 0$ otherwise. The associated density profile is thus given by:

$$\rho(\psi) = \hat{A}\hat{\rho}(\psi) = \hat{A} e^{\psi} \gamma(5/2, \psi) \tag{4}$$

where $\psi = a\{H_0 - [\Phi_C + \Phi_T]\}$ is the dimensionless escape energy.

An equivalent physical model and choice for the distribution function is considered also in [3] and [4].

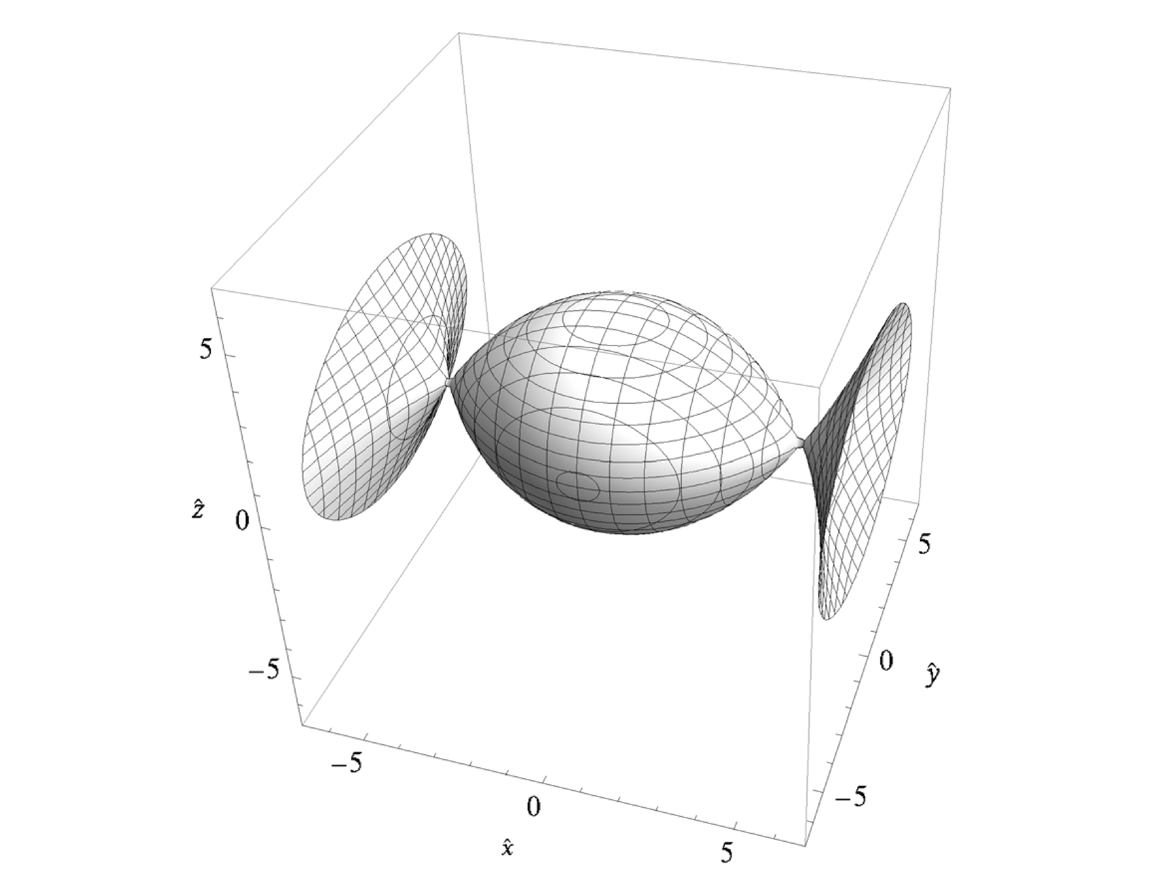


Figure 1 The boundary of the model is defined as the relevant zero-velocity surface given by $\psi(\mathbf{r}) = 0$. Here a 2nd order critical model with $\Psi = 2$ and $\epsilon = 7.043 \times 10^{-4}$ is displayed; the galactic potential is Keplerian ($\nu = 3$).

2. The Parameter Space

The models are characterized by:

- two physical scales (i.e., free constants A and a)
- two dimensionless parameters

$$\textbf{Concentration} \quad \leftrightarrow \quad \Psi \equiv \psi(\mathbf{0}) \tag{5}$$

$$\textbf{Tidal strength} \quad \leftrightarrow \quad \epsilon \equiv \frac{\Omega^2}{4\pi G \rho_0} \tag{6}$$

For a given value of the central potential well Ψ, there exists a (maximum) critical value for the tidal strength parameter. **Two tidal regimes exist** (i.e., models with small / significant departures from spherical symmetry); the models studied in [4] correspond to our "critical" models.

3. The mathematical problem

Models are constructed by solving the Poisson equation in dimensionless form (with $r_0 = \sqrt{9/(4\pi G \rho_0 a)}$ as scale length):

$$\hat{\nabla}^2 \psi = -9 \left[\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)} + \epsilon(1-\nu) \right] \, , \tag{7}$$

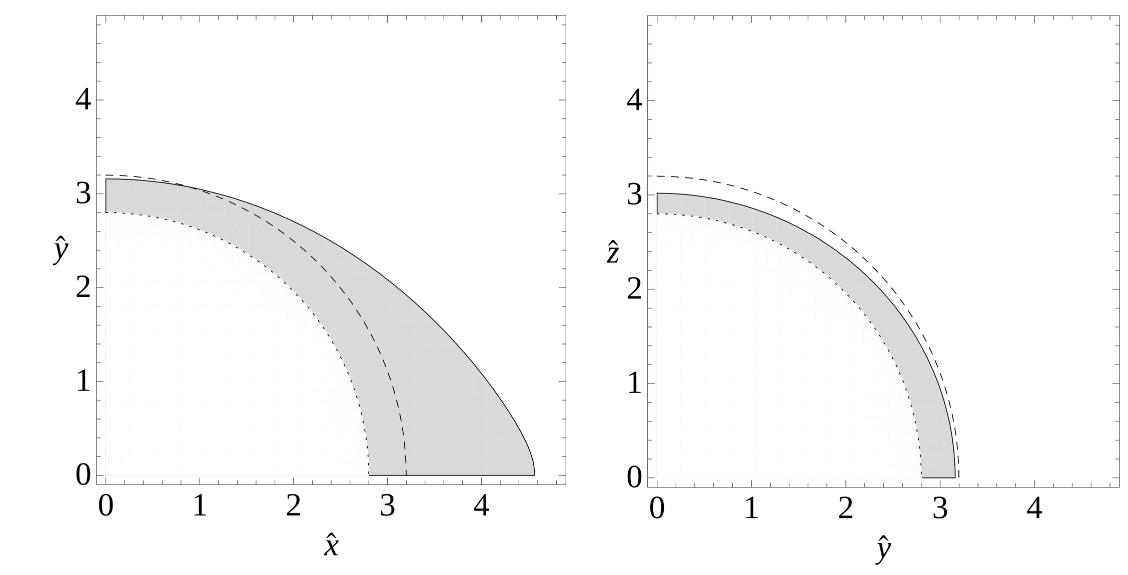
with the requirement of finiteness (Ψ) and regularity of the solution at the origin. For negative values of ψ we should refer to:

$$\hat{\nabla}^2 \psi = -9\epsilon(1-\nu) \, , \tag{8}$$

i.e. the Laplace equation for Φ_C, requiring that, at large radii, the natural behavior $a\Phi_C \to 0$ is respected.
Poisson (internal) and Laplace (external) domains are thus separated by the boundary surface which is unknown *a priori*: we have to solve an **elliptic partial differential equation in a free boundary problem**.

4. Solution in terms of matched asymptotic expansions

- The solution in the internal and external domains are expressed as an **asymptotic series** with respect to ϵ, with general term $\psi_k(\hat{\mathbf{r}})$.
- We use the method of matched asymptotic expansions [5], in order to obtain a uniform solution across the separation free surface.
- Note that the **validity** of the expansions **breaks down** where the second term of each of the two series becomes comparable to the first, i.e where $\psi_0 = \mathcal{O}(\epsilon)$; this region can be considered as a **boundary layer** (see Fig. 2).
- This method of solution is basically the same as the one proposed for the analogous mathematical problem that arises in the determination of the structure of rigidly rotating fluid polytropes [6].
- By introducing an expansion in (real) orthonormal spherical harmonics, we calculated the full explicit solution to **2 orders** in ϵ. The relevant PDEs for $\psi_k(\hat{\mathbf{r}})$ are thus reduced to sets of simple (radial) ODEs for which a numerical solution is required only in the internal domain (we used a fourth-order Runge-Kutta code).
- $\psi^{(1)} \leftrightarrow$ harmonics with $l = 0, 2$, $m \geq 0$ and even
  $\psi^{(2)} \leftrightarrow$ harmonics with $l = 0, 2, 4$, $m \geq 0$ and even
  From the structure of the relevant equations, we prove, *by induction*, that the k-th order solution $\psi^{(k)}$ contains only the $l = 0, 2, .., 2k$ harmonics.



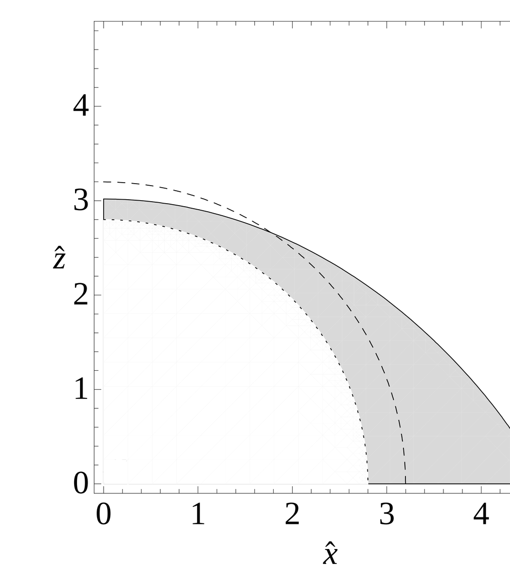


Figure 2 Sections in the three coordinate planes for a 2nd order model with $\Psi = 2$ and $\epsilon = 7.000 \times 10^{-4}$ (almost critical regime, see Fig.1).
The boundary surfaces of the triaxial model (solid), of the internal region (dotted), and of the corresponding spherical King model (dashed) are represented; the shaded area corresponds to the boundary layer.

5. Extension to the case of internal rotation

In the absence of external tides but in the presence of non-vanishing total angular momentum, general statistical arguments [7] suggest that, in the Maxwell-Boltzmann distribution function, one should replace the energy E with the quantity $H = E - J_z\omega$, where ω turns out to represent the (solid body) angular velocity of the system.
Extension of spherical King models to the case of internal rigid rotation can thus be performed with the same method that we used to study the effects of external tides. Note that these **axisymmetric** models differ from those discussed in [8]. The parameter space is equivalent to the one presented in the tidal case, with:

$$\textbf{Rotation strength} \quad \leftrightarrow \quad \chi = \frac{\omega^2}{4\pi G \rho_0} \, , \tag{9}$$

playing the role of ϵ.

6. Conclusions

- Models of quasi-relaxed triaxial stellar systems in which the shape is due to the presence of external tides have been constructed; they reduce to standard spherical King models in the absence of the tidal field.
- We considered two additional alternative methods of solution for this mathematical problem: the first is based on iteration seeded by the spherical solution, while the second is an application of the method of strained coordinates (inspired by [9]).
- The same procedure has been used to extend other isotropic truncated models to the triaxial case (we studied the case of polytropes of index $1 < n < 5$).
- Models of quasi-relaxed stellar systems flattened by (rigid) rotation have also been constructed.
- A thorough description of the family of tidal models in terms of intrinsic and projected quantities can be found in the poster "Properties of quasi-relaxed stellar systems in an external tidal field" (Varri & Bertin).

References

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