

THE CONSTRUCTION OF NONSPHERICAL MODELS OF QUASI-RELAXED STELLAR SYSTEMS

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ABSTRACT

Spherical models of collisionless but quasi-relaxed stellar systems have long been studied as a natural framework for the description of globular clusters. Here we consider the construction of self-consistent models under the same physical conditions, but including explicitly the ingredients that lead to departures from spherical symmetry. In particular, we focus on the effects of the tidal field associated with the hosting galaxy. We then take a stellar system on a circular orbit inside a galaxy represented as a “frozen” external field. The equilibrium distribution function is obtained from the one describing the spherical case by replacing the energy integral with the relevant Jacobi integral in the presence of the external tidal field. Then the construction of the model requires the investigation of a singular perturbation problem for an elliptic partial differential equation with a free boundary, for which we provide a method of solution to any desired order, with explicit solutions to 2 orders. We outline the relevant parameter space, thus opening the way to a systematic study of the properties of a two-parameter family of physically justified nonspherical models of quasi-relaxed stellar systems. The general method developed here can also be used to construct models for which the nonspherical shape is due to internal rotation. Eventually, the models will be a useful tool to investigate whether the shapes of globular clusters are primarily determined by internal rotation, by external tides, or by pressure anisotropy.

Subject headings: globular clusters: general — methods: analytical — stellar dynamics

1. INTRODUCTION

Large stellar systems can be studied as collisionless systems, by means of a one-star distribution function obeying the combined set of the collisionless Boltzmann equation and the Poisson equation, under the action of the self-consistent mean potential. For elliptical galaxies the relevant two-star relaxation times do actually exceed their age; an imprint of partial relaxation may be left at the time of their formation (if we refer to a picture of formation via incomplete violent relaxation; Lynden-Bell 1967; van Albada 1982), but otherwise they should be thought of as truly collisionless systems, generally characterized by an anisotropic pressure tensor. In turn, for globular clusters the relevant relaxation times are typically shorter than their age, so that we may argue that for many of them the two-star relaxation processes have had enough time to act and to bring them close to a thermodynamically relaxed state, with their distribution function close to a Maxwellian. This line of arguments has led to the development of well-known dynamical models for globular clusters (King 1965, 1966).

King models are based on a quasi-Maxwellian isotropic distribution function $f_K(E)$ in which a truncation prescription, continuous in phase space, is set heuristically to incorporate the presence of external tides; but otherwise, they are fully self-consistent (i.e., no external fields are actually considered) and perfectly spherical. Empirically, the simplification of spherical symmetry is encouraged by the fact that in general globular clusters have a round appearance. Indeed, as a zeroth-order description, these models have had remarkable success in applications to observed globular clusters (e.g., see Spitzer 1987; Djorgovski & Meylan 1994; McLaughlin & van der Marel 2005 and references therein). In recent years, great progress has been made in the acquisition of detailed quantitative information about the structure of these stellar systems, especially in relation to the measurement of the proper motions of thousands of individual stars (see van Leeuwen et al. 2000; McLaughlin et al. 2006), with the possibility of getting a direct five-dimensional view of their phase space. Such progress

calls for renewed efforts on the side of modeling. More general models would allow us to address the issue of the origin of the observed departures from spherical symmetry. In fact, it remains to be established which physical ingredient among rotation, pressure anisotropy, and tides is the primary cause of the flattening of globular clusters (e.g., see King 1961; Frenk & Fall 1982; Geyer et al. 1983; White & Shawl 1987; Davoust & Prugniel 1990; Han & Ryden 1994; Ryden 1996; Goodwin 1997; van den Bergh 2008 and references therein)

As in the case of the study of elliptical galaxies (e.g., see Bertin & Stiavelli 1993 and references therein), different approaches can be taken to the construction of models. Broadly speaking, two complementary paths can be followed. In the first, “descriptive” approach, under suitable geometrical (on the intrinsic shape) and dynamical (e.g., on the absence or presence of dark matter) hypotheses, the available data for an individual stellar system are imposed as constraints to derive the internal orbital structure (distribution function) most likely to correspond to the observations. This approach is often carried out in terms of codes that generalize a method introduced by Schwarzschild (1979); for an application to the globular cluster ω Cen, see van de Ven et al. (2006). In the second, “predictive” approach, one proposes a formation/evolution scenario in order to identify a physically justified distribution function for a wide class of objects, and then one proceeds to investigate, by comparison with observations of several individual objects, whether the data support the general physical picture that has been proposed. Indeed, King models belong to this latter approach.

The purpose of this paper is to extend the description of quasi-relaxed stellar systems, so far basically limited to the spherical King models, to the nonspherical case. There are at least three different ways of extending spherical isotropic models of quasi-relaxed stellar systems (such as King models), by modifying the distribution function so as to include: (1) the explicit presence of a nonspherical tidal field, (2) the presence of internal rotation, or (3) the presence of some pressure anisotropy. As noted above, these correspond to the physical ingredients that, separately, may

be thought to be at the origin of the observed nonspherical shapes. We thus focus on the construction of physically justified models, as an extension of the King models, in the presence of external tides and, briefly, on the extension of the models to the presence of rigid internal rotation. A first-order analysis of the triaxial tidal problem addressed in this paper was carried out by Heggie & Ramamani (1995), and the effect of a “frozen” tidal field on (initially) spherical King models was studied by Weinberg (1993) using N -body simulations. For the axisymmetric problem associated with the presence of internal rotation, a first-order analysis of a simply truncated Maxwellian distribution perturbed by a rigid rotation was given by Kormendy & Anand (1971); different models where differential rotation is considered were proposed by Prendergast & Tomer (1970) and by Wilson (1975), also in view of extensions to the presence of pressure anisotropy, which goes beyond the scope of this paper. Models that represent a direct generalization of the King family to the case with differential rotation have also been examined, with particular attention to their thermodynamic properties (Lagoute & Longaretti 1996; Longaretti & Lagoute 1996). In principle, the method of solution that we present below can deal with the extension of other spherical isotropic models with finite size that are not of the King form (e.g., see Woolley & Dickens 1962; Davoust 1977; see also the interesting suggestion by Madsen 1996).

The study of self-consistent collisionless equilibrium models has a long tradition not only in stellar dynamics, but also in plasma physics (e.g., see Harris 1962; Attico & Pegoraro 1999). We note that in both research areas, a study in the presence of external fields, especially when the external field is bound to break the natural symmetry associated with the one-component problem, is only rarely considered.

The paper is organized as follows. Section 2 introduces the reference physical model, in which a globular cluster is imagined to move on a circular orbit inside a host galaxy treated as a frozen background field; the modified distribution function for such a cluster is then identified and the relevant parameter space defined. In § 3 we set up the mathematical problem associated with the construction of the related self-consistent models. For models generated by the spherical $f_K(E)$, § 4 and Appendix A give the complete solution in terms of matched asymptotic expansions. Alternative methods of solution are briefly discussed in § 5. The concluding § 6 gives a summary of the paper, with a short discussion of the results obtained. In Appendix B we show how the method developed in this paper can be applied to construct quasi-relaxed models flattened by rotation in the absence of external tides. In Appendix C we show how the method can be applied to other isotropic truncated models, different from King models.

Technically, the mathematical problem of a singular perturbation with a free boundary that is faced here is very similar to the problem noted in the theory of rotating stars, starting with Milne (see Tassoul 1978; Milne 1923; Chandrasekhar 1933; Krogdahl 1942; Chandrasekhar & Lebovitz 1962; Monaghan & Roxburgh 1965). The problem was initially dealt with using inadequate tools; a satisfactory solution of the singular perturbation problem was obtained only later, by Smith (1975, 1976).

2. THE PHYSICAL MODEL

2.1. The Tidal Potential

As a reference case, we consider an idealized model in which the center of mass of a globular cluster is imagined to move on a circular orbit of radius R_0 , characterized by orbital frequency Ω , inside a host galaxy. For simplicity, we focus on the motion of the stars inside the globular cluster and model the galaxy, taken to

have very large mass, by means of a *frozen* gravitational field (which we call the galactic field, described by the potential Φ_G), with a given overall symmetry. This choice makes us ignore interesting effects that are generally present in the full interaction between a “satellite” and a galaxy; in a sense, we are taking a complementary view of an extremely complex dynamical situation, with respect to other investigations, such as those that lead to a discussion of the mechanism of dynamical friction (in which the globular cluster or satellite is modeled as a rigid body and the stars of the galaxy are taken as the “live” component; see Bontekoe & van Albada 1987; Bertin et al. 2003; Arena & Bertin 2007; and references therein). Therefore, we will be initially following the picture of a *restricted three-body problem*, with one important difference, that the “secondary” is not treated as a point mass but as a “live” stellar system, described by the cluster mean field potential Φ_C . In this extremely simple orbital choice for the cluster center of mass, in the corotating frame the associated tidal field is time independent, so we can proceed to the construction of a stationary dynamical model.

We consider the galactic potential Φ_G to be spherically symmetric, i.e., $\Phi_G = \Phi_G(R)$, with $R = (X^2 + Y^2 + Z^2)^{1/2}$, in terms of a standard set of Cartesian coordinates (X, Y, Z) , so that $\Omega^2 = [d\Phi_G(R)/dR]_{R_0}/R_0$. Let (X, Y) be the orbit plane of the center of mass of the cluster. We then introduce a local rotating frame of reference, so that the position vector is given by $\mathbf{r} = (x, y, z)$, with the origin in the center of mass of the cluster and for which the x -axis points away from the center of the galactic field, the y -axis follows the direction of the cluster rotation in its orbit around the galaxy, and the z -axis is perpendicular to the orbit plane (according to the right-hand rule). In such a rotating local frame, the relevant Lagrangian, describing the motion of a star belonging to the cluster, is (cf. Chandrasekhar 1942, eq. [5.510])

$$\mathcal{L} = \frac{1}{2} \left\{ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \Omega^2 [(R_0 + x)^2 + y^2] + 2\Omega(R_0 + x)\dot{y} - 2\Omega\dot{x}y \right\} - \Phi_G(R) - \Phi_C(x, y, z), \quad (1)$$

where $R = [(R_0 + x)^2 + y^2 + z^2]^{1/2}$ and the terms responsible for centrifugal and Coriolis forces are explicitly displayed.

If we suppose that the size of the cluster is small compared to R_0 , we can adequately represent the galactic field by its linear approximation with respect to the local coordinates (the so-called tidal approximation). The corresponding equations of motion for a single star in the rotating local frame are

$$\ddot{x} - 2\Omega\dot{y} - (4\Omega^2 - \kappa^2)x = -\frac{\partial\Phi_C}{\partial x}, \quad (2)$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Phi_C}{\partial y}, \quad (3)$$

$$\ddot{z} + \Omega^2 z = -\frac{\partial\Phi_C}{\partial z}, \quad (4)$$

where κ is the epicyclic frequency at R_0 , given by $\kappa^2 = 3\Omega^2 + (d^2\Phi_G/dR^2)_{R_0}$. Note that the assumed symmetry for Φ_G introduces a cancellation between the kinematic term $y\Omega^2$ and the gradient of the galactic potential $\partial\Phi_G/\partial y$ and makes the vertical acceleration $-\partial\Phi_G/\partial z$ approximately equal to $-z\Omega^2$.

These equations admit an energy (isolating) integral of the motion, known as the Jacobi integral,

$$H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_T + \Phi_C, \quad (5)$$

where

$$\Phi_T = \frac{1}{2}\Omega^2(z^2 - \nu x^2) \quad (6)$$

is the tidal potential. Here, $\nu \equiv 4 - \kappa^2/\Omega^2$ is a generally positive dimensionless coefficient.

Thus, at the level of single-star orbits, we note that, in general, the tidal potential leads to a compression in the z -direction, a stretching in the x -direction, and leaves the y -direction untouched. The tidal potential is static, breaks the spherical symmetry, but is characterized by reflection symmetry with respect to the three natural coordinate planes; strictly speaking, the symmetry with respect to (y, z) is applicable only in the limit of an infinitely massive host galaxy (see Spitzer 1987). In turn, we see that the geometry of the tidal potential induces a nonspherical distortion of the cluster shape collectively, in particular, an elongation along the x -axis and a compression along the z -axis. In practice, the numerical coefficient ν that defines quantitatively the induced distortion depends on the galactic potential. We recall that we have $\nu = 3$ for a Keplerian potential, $\nu = 2$ for a logarithmic potential, while for a Plummer model the dimensionless coefficient depends on the location of the circular orbit with respect to the model scale radius b , with $\nu(R_0) = 3R_0^2/(b^2 + R_0^2)$ (for a definition of the Plummer model see, e.g., Bertin 2000).

Different assumptions on the geometry of the galactic field can be treated with tools similar to those developed here, leading to a similar structure of the equations of motion, with a slight modification of the tidal field. In particular, for an axisymmetric galactic field, the tidal potential differs from the one obtained here only by the z -term (Chandrasekhar 1942; Heggie & Hut 2003). This case is often considered, for example, by referring to a globular cluster in a circular orbit on the (axisymmetric) disk of our Galaxy (see Heggie & Ramamani 1995; Ernst et al. 2008), for which Φ_T is then formulated in terms of the Oort constants.

In the physical model outlined in this section, the typical dynamical time associated with the star orbits inside the cluster is much smaller than the (external) orbital time associated with Ω . Therefore, in an asymptotic sense, the equilibrium configurations that we construct in the rest of the paper can actually be generalized, with due qualifications, to more general orbits of the cluster inside a galaxy, provided we interpret the results that we are going to obtain as applicable only to a small piece of the cluster orbit.

2.2. The Distribution Function

As outlined in § 1, we wish to extend the description of quasi-relaxed stellar systems [so far, basically limited to spherical models associated with distribution functions $f = f(E)$, dependent only on the single-star specific energy $E = v^2/2 + \Phi_C$] to the nonspherical case, by including the presence of a nonspherical tidal field explicitly. Given the success of the spherical King models in the study of globular clusters, we focus on the extension of models based on $f_K(E)$, which is defined as a “lowered” Maxwellian, continuous in phase space, with an energy cutoff which implies the existence of a boundary at the truncation radius r_{tr} .

Therefore, we consider (partially) self-consistent models characterized by the distribution function

$$f_K(H) = A[\exp(-aH) - \exp(-aH_0)] \quad (7)$$

if $H \leq H_0$ and $f_K(H) = 0$ otherwise, in terms of the Jacobi integral defined by equation (5). Here H_0 is the cutoff value for the Jacobi integral, while A and a are positive constants.

In velocity space, the inequality $H \leq H_0$ identifies a spherical region given by $0 \leq v^2 \leq 2\psi(\mathbf{r})/a$, where

$$\psi(\mathbf{r}) = a\{H_0 - [\Phi_C(\mathbf{r}) + \Phi_T(x, z)]\} \quad (8)$$

is the dimensionless escape energy. Therefore, the boundary of the cluster is defined as the relevant zero-velocity surface by the condition $\psi(\mathbf{r}) = 0$ and is given only implicitly by an equipotential (*Hill*) surface for the total potential $\Phi_C + \Phi_T$; in fact, its geometry depends on the properties of the tidal potential (of known characteristics; see eq. [6]) and of the cluster potential (unknown a priori, to be determined as the solution of the associated Poisson equation).

The value of the cutoff potential H_0 should be chosen in such a way that the surface that defines the boundary is closed. The last (i.e., outermost) closed Hill surface is a *critical* surface, because it contains two saddle points that represent the Lagrangian points of the restricted three-body problem outlined in § 2.1. From equations (2)–(4) we see that such Lagrangian points are located symmetrically with respect to the origin of the local frame of reference and lie on the x -axis. Their distance from the origin is called the *tidal radius*, which we denote by r_T , and can be determined from the condition

$$\frac{\partial\psi}{\partial x}(r_T, 0, 0) = 0. \quad (9)$$

If, as a zeroth-order approximation, we use a simple Keplerian potential for the cluster potential Φ_C , we recover the classical expression (e.g., see Spitzer 1987)

$$r_T^{(0)} = \left(\frac{GM}{\Omega^2\nu}\right)^{1/3}, \quad (10)$$

where M is the total mass of the cluster.

As for the spherical King model, the density profile associated with the distribution function from equation (7) is given by

$$\rho(\psi) = \hat{A}e^{\psi}\gamma\left(\frac{5}{2}, \psi\right) \equiv \hat{A}\hat{\rho}(\psi), \quad (11)$$

where $\hat{A} = 8\pi A\sqrt{2}e^{-aH_0}/(3a^{3/2})$. We recall that the incomplete gamma function has a nonnegative real value only in correspondence to a nonnegative argument. In the following we denote the central density of the cluster by $\rho_0 = \hat{A}\hat{\rho}(\Psi)$, where $\Psi \equiv \psi(\mathbf{0})$ is the depth of the central potential well.

2.3. The Parameter Space

The models defined by $f_K(H)$ are characterized by two physical scales (e.g., the two free constants A and a or, correspondingly, the total mass M and the central density ρ_0 of the cluster) and two dimensionless parameters. The first dimensionless parameter can be defined, as in the spherical King models, to measure the *concentration* of the cluster. We can thus consider the quantity Ψ , introduced at the end of § 2.2, or we may refer to the commonly used concentration parameter,

$$C = \log(r_{\text{tr}}/r_0), \quad (12)$$

where $r_0 = [9/(4\pi G\rho_0 a)]^{1/2}$ is a scale length related to the size of the core and r_{tr} is the truncation radius of the spherical King model associated with the same value of the central potential well Ψ (the relation between C and Ψ is one to one; e.g., see Bertin 2000).

The second dimensionless parameter characterizes the strength of the (external) tidal field,

$$\epsilon \equiv \frac{\Omega^2}{4\pi G\rho_0}. \quad (13)$$

The definition arises naturally from the dimensionless formulation of the Poisson equation that describes the (partially) self-consistent problem (to be addressed in § 3).

In principle, for a given choice of the dimensional scales (A and a) the truncation radius or the concentration parameter of a spherical King model can be set arbitrarily. In practice, the physical motivation of the models suggests that the truncation radius r_{tr} should be taken to be of the order of (and not exceed) the tidal radius r_T , introduced in § 2.2 (see eq. [9]). We may thus define an *extension parameter* as the ratio between the truncation radius of the corresponding spherical model and the tidal radius r_T ,

$$\delta \equiv \frac{r_{\text{tr}}}{r_T}. \quad (14)$$

For a given value of the central potential well Ψ , there exists a (maximum) critical value for the tidal strength parameter, which we denote by ϵ_{cr} , corresponding to the maximum value for the extension parameter δ_{cr} , which can be found by solving the system

$$\begin{aligned} \frac{\partial\psi}{\partial x}(r_T, 0, 0; \epsilon_{\text{cr}}) &= 0, \\ \psi(r_T, 0, 0; \epsilon_{\text{cr}}) &= 0. \end{aligned} \quad (15)$$

From this system, if we use the zeroth-order Keplerian approximation for Φ_C , we find that $\delta_{\text{cr}}^{(0)} = 2/3$ (see Spitzer 1987).

For our two-parameter family of models we thus expect two tidal regimes to exist. For models characterized by the pairs (Ψ, ϵ) near the critical condition $\delta \approx \delta_{\text{cr}}$ the tidal distortion should be maximal, while for models with pairs well below criticality, only small departures from spherical symmetry should occur. A thorough exploration of the parameter space will be carried out in a separate paper (A. L. Varri & G. Bertin, in preparation). In closing, we note that the models proposed and studied by Heggie & Ramamani (1995) correspond to the pairs in parameter space that we have called critical.

3. THE MATHEMATICAL PROBLEM

The (partially) self-consistent models associated with the distribution function defined by equation (7) are constructed by solving the relevant Poisson equation. In terms of the dimensionless escape energy ψ , given by equation (8), the Poisson equation (for $\psi \geq 0$) can be written as

$$\nabla^2(\psi + a\Phi_T) = -\frac{9}{r_0^2} \frac{\rho}{\rho_0} = -\frac{9}{r_0^2} \frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)}, \quad (16)$$

where r_0 is the scale length introduced in § 2.3 (see eq. [12]). We then rescale the coordinates and introduce the dimensionless position vector $\hat{\mathbf{r}} = \mathbf{r}/r_0$, so that $\hat{\nabla}^2 = r_0^2 \nabla^2$ and $a\Phi_T \equiv \epsilon T = 9\epsilon(\hat{z}^2 - \nu\hat{x}^2)/2$, where we have made use of the tidal parameter introduced in equation (13). Therefore, the Poisson equation, for $\psi \geq 0$, can be written in dimensionless form as

$$\hat{\nabla}^2\psi = -9 \left[\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)} + \epsilon(1 - \nu) \right], \quad (17)$$

while for negative values of ψ we should refer to

$$\hat{\nabla}^2\psi = -9\epsilon(1 - \nu), \quad (18)$$

i.e., the Laplace equation $\hat{\nabla}^2(a\Phi_C) = 0$.

The mathematical problem is completed by specifying the appropriate boundary conditions. As for the spherical King models, we require regularity of the solution at the origin,

$$\psi(\mathbf{0}) = \Psi, \quad (19)$$

$$\hat{\nabla}\psi(\mathbf{0}) = \mathbf{0}, \quad (20)$$

and at large radii,

$$\psi + \epsilon T \rightarrow aH_0, \quad (21)$$

which corresponds to $a\Phi_C \rightarrow 0$.

Poisson and Laplace domains are thus separated by the surface defined by $\psi = 0$, which is unknown a priori. In other words, we have to solve an elliptic partial differential equation in a *free boundary* problem.

In the ordinary differential problem that characterizes the construction of spherical models with finite mass, the condition of vanishing cluster potential at large radii (together with the regularity conditions at the origin) overdetermines the problem, which can then be seen as an eigenvalue problem (e.g., see § 2.5 in Bertin & Stiavelli 1993). Indeed, for the King models the integration of the Poisson equation from the origin outward, with “initial conditions” from equations (19)–(20), sets the relation between the ratio r_{tr}/r_0 and Ψ in order to meet the requirement from equation (21), with r_{tr}/r_0 thus playing the role of an “eigenvalue.”

In the more complex, three-dimensional situation that we are facing here, the existence of two different domains, internal (Poisson) and external (Laplace), suggests the use of the method of matched asymptotic expansions in order to obtain a uniform solution across the separation free surface. The solutions in the internal and external domains are expressed as asymptotic series with respect to the tidal parameter ϵ , which is assumed to be small (following the physical model described in § 2.2),

$$\psi^{(\text{int})}(\hat{\mathbf{r}}; \epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} \psi_k^{(\text{int})}(\hat{\mathbf{r}}) \epsilon^k, \quad (22)$$

$$\psi^{(\text{ext})}(\hat{\mathbf{r}}; \epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} \psi_k^{(\text{ext})}(\hat{\mathbf{r}}) \epsilon^k, \quad (23)$$

with spherical symmetry assumed for the zeroth-order terms. The internal solution should obey the boundary conditions from equations (19)–(20), while the external solution should satisfy equation (21). The two representations should be properly connected at the surface of the cluster.

On the other hand, for any small but finite value of ϵ the boundary, defined by $\psi = 0$, will be different from the unperturbed boundary, defined by $\psi_0 = 0$, so that, for each of the two representations given above, there will be a small region in the vicinity of the surface of the cluster where the leading term is vanishingly small and actually smaller than the remaining terms of the formal series. Therefore, we expect the validity of the expansion to break down where the second term becomes comparable to the first, i.e., where $\psi_0 = O(\epsilon)$. This region can be considered as a *boundary layer*, which should be examined in “microscopic” detail by a suitable rescaling of the spatial coordinates and for

which an adequate solution $\psi^{(\text{lay})}$, expressed as a different asymptotic series, should be constructed. To obtain a uniformly valid solution over the entire space, an asymptotic matching is performed between the pairs $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ and $(\psi^{(\text{lay})}, \psi^{(\text{ext})})$, thus leading to a solution ψ , obeying all the desired boundary conditions, in terms of three different, but matched, representations. This method of solution is basically the same method proposed by Smith (1975) for the analogous mathematical problem that arises in the determination of the structure of rigidly rotating fluid polytropes.

4. SOLUTION IN TERMS OF MATCHED ASYMPTOTIC EXPANSIONS

The complete solution to 2 significant orders in the tidal parameter is now presented. The formal solution to 3 orders is also displayed because of the requirements of the Van Dyke principle of asymptotic matching (cf. Van Dyke 1975, eq. [5.24]) that we have adopted.

4.1. Internal Region

If we insert the series from equation (22) in the Poisson equation (17), under the conditions from equations (19)–(20), we obtain an (infinite) set of Cauchy problems for ψ_k . The problem for the zeroth-order term (i.e., the unperturbed problem with $\epsilon = 0$) is the one defining the construction of the spherical and fully self-consistent King models,

$$\psi_0^{(\text{int})''} + \frac{2}{\hat{r}} \psi_0^{(\text{int})'} = -9 \frac{\hat{\rho}(\psi_0^{(\text{int})})}{\hat{\rho}(\Psi)}, \quad (24)$$

with $\psi_0^{(\text{int})}(0) = \Psi$ and $\psi_0^{(\text{int})'}(0) = 0$, where the prime denotes a derivative with respect to the argument \hat{r} . We recall that the truncation radius \hat{r}_{tr} , which defines the boundary of the spherical models, is given implicitly by $\psi_0^{(\text{int})}(\hat{r}_{\text{tr}}) = 0$.

Let us introduce the quantities

$$R_j(\hat{r}; \Psi) \equiv \frac{9}{\hat{\rho}(\Psi)} \frac{d^j \hat{\rho}}{d\psi^j} \Big|_{\psi_0^{(\text{int})}}. \quad (25)$$

These quantities depend on \hat{r} implicitly, through the function $\psi_0^{(\text{int})}$; in turn, the dependence on Ψ is both explicit [through the term $\hat{\rho}(\Psi)$] and implicit [because the function $\psi_0^{(\text{int})}(\hat{r})$ depends on the value of Ψ]. For convenience, we give the expression of the first terms of the sequence (cf. eq. [11]), $R_1 = [9/\hat{\rho}(\Psi)][\hat{\rho}(\psi_0^{(\text{int})}) + (\psi_0^{(\text{int})})^{3/2}]$, $R_2 = R_1 + 27(\psi_0^{(\text{int})})^{1/2}/[2\hat{\rho}(\Psi)]$, and $R_3 = R_2 + 27(\psi_0^{(\text{int})})^{-1/2}/[4\hat{\rho}(\Psi)]$. Note that for $\psi_0^{(\text{int})} \rightarrow 0$, i.e., for $\hat{r} \rightarrow \hat{r}_{\text{tr}}$, $R_1 \rightarrow 0$, $R_2 \rightarrow 0$, while for $j \geq 3$, the quantities R_j actually *diverge*. This is one more indication of the singular character of our perturbation analysis, which brings in some fractional power dependence on the perturbation parameter ϵ (see also the expansion from eq. [46]).

Therefore, the equations governing the next two orders (for $\psi_k^{(\text{int})}$ with $k = 1, 2$) can be written as:

$$[\hat{\nabla}^2 + R_1(\hat{r}; \Psi)] \psi_1^{(\text{int})} = -9(1 - \nu), \quad (26)$$

$$[\hat{\nabla}^2 + R_1(\hat{r}; \Psi)] \psi_2^{(\text{int})} = -R_2(\hat{r}; \Psi) (\psi_1^{(\text{int})})^2, \quad (27)$$

with $\psi_1^{(\text{int})}(\mathbf{0}) = \psi_2^{(\text{int})}(\mathbf{0}) = 0$ and $\hat{\nabla} \psi_1^{(\text{int})}(\mathbf{0}) = \hat{\nabla} \psi_2^{(\text{int})}(\mathbf{0}) = \mathbf{0}$. The equation for $k = 3$ is recorded in Appendix A.1, where we also describe the structure of the general equation for $\psi_k^{(\text{int})}$.

For any given order of the expansion, the operator acting on the function $\psi_k^{(\text{int})}$ (see the left-hand side of eqs. [26] and [27]) is the same, i.e., a Laplacian “shifted” by the function $R_1(\hat{r}; \Psi)$. If we thus expand every term $\psi_k(\hat{\mathbf{r}})$ in spherical harmonics,¹

$$\psi_k^{(\text{int})}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \psi_{k,lm}^{(\text{int})}(\hat{r}) Y_{lm}(\theta, \phi), \quad (28)$$

the three-dimensional differential problem is reduced to a one-dimensional (radial) problem, characterized by the following second-order, linear ordinary differential operator,

$$\mathcal{D}_l = \frac{d^2}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{d}{d\hat{r}} - \frac{l(l+1)}{\hat{r}^2} + R_1(\hat{r}; \Psi). \quad (29)$$

In general, for a fixed value of l , two independent solutions to the homogeneous problem $\mathcal{D}_l f = 0$ are expected² to behave like \hat{r}^l and $1/\hat{r}^{l+1}$ for $\hat{r} \rightarrow 0$. Because of the presence of $R_1(\hat{r}; \Psi)$, solutions to equations where \mathcal{D}_l appears have to be obtained numerically.

For $k = 1$ (see eq. [26]) we thus have to address the following problem. For $l = 0$, the relevant equation is

$$\mathcal{D}_0 f_{00} = -9(1 - \nu), \quad (30)$$

where $f_{00} \equiv \psi_{1,00}^{(\text{int})}/(4\pi)^{1/2}$, with $f_{00}(0) = f_{00}'(0) = 0$. Here we do not have to worry about including solutions to the associated homogeneous problem, because one of the two independent solutions would be singular at the origin and the other would be forced to vanish by the required condition at $\hat{r} = 0$. For $l \geq 1$ we have

$$\mathcal{D}_l \psi_{1,lm}^{(\text{int})} = 0, \quad (31)$$

with $\psi_{1,lm}^{(\text{int})}(0) = \psi_{1,lm}^{(\text{int})'}(0) = 0$. Both equation (31) and the associated boundary conditions are homogeneous. Therefore, the solution is undetermined by an m -dependent multiplicative constant, $\psi_{1,lm}^{(\text{int})}(\hat{r}) = A_{lm} \gamma_l(\hat{r})$, with $\gamma_l(\hat{r}) \sim \hat{r}^l$ for $\hat{r} \rightarrow 0$ (the singular solution is excluded by the boundary conditions at the origin). Then the complete formal solution is

$$\psi_1^{(\text{int})}(\hat{\mathbf{r}}) = f_{00}(\hat{r}) + \sum_{l=1}^{\infty} \sum_{m=-l}^l A_{lm} \gamma_l(\hat{r}) Y_{lm}(\theta, \phi), \quad (32)$$

where the constants are ready to be determined by means of the asymptotic matching with $\psi_1^{(\text{lay})}(\hat{\mathbf{r}})$ at the boundary layer.

For $k = 2$ (see eq. [27]) the relevant equations are

$$\mathcal{D}_l \psi_{2,lm}^{(\text{int})} = -R_2(\hat{r}; \Psi) (\psi_1^{(\text{int})})_{lm}^2, \quad (33)$$

where on the right-hand side the function $\psi_1^{(\text{int})}$ is that derived from the solution of the first-order problem (which shows the progressive character of this method for the construction of solutions). In Appendix A.2 the equations for the six relevant harmonics are

¹ We use orthonormalized real spherical harmonics with the Condon-Shortley phase; with respect to the toroidal angle ϕ , they are even for $m \geq 0$ and odd otherwise.

² We note that $R_1(0, \Psi) = 9[1 + \Psi^{3/2}/\hat{\rho}(\Psi)]$, i.e., a numerical positive constant. Therefore, for $\hat{r} \rightarrow 0$ the operator \mathcal{D}_l tends to the operator associated with the spherical Bessel functions of the first and of the second kind (e.g., see Abramowitz & Stegun 1964, eq. [10.1.1] for the equation and eqs. [10.1.4] and [10.1.5] for the limiting values of the functions for small arguments).

displayed explicitly. The boundary conditions to be imposed at the origin are again homogeneous, $\psi_{2,lm}^{(int)}(0) = \psi_{2,lm}^{(int)'}(0) = 0$. For a fixed harmonic (l, m) with $l > 0$, the general solution of equation (33) is the sum of a particular solution [which we denote by $g_{lm}(\hat{r})$] and of a regular solution to the associated homogeneous problem given by equation (31) [which we call $B_{lm}\gamma_l(\hat{r})$, with $\gamma_l(\hat{r})$ being the same functions introduced for the first-order problem]. Obviously, the particular solution exists only when equation (33) is nonhomogeneous, i.e., only for those values of (l, m) that correspond to a nonvanishing coefficient in the expansion of $(\psi_1^{(int)})^2$ in spherical harmonics. As noted in the first-order problem ($k = 1$), the associated homogeneous problem for $l = 0$ has no nontrivial solution. Then we can express the complete solution as

$$\psi_2^{(int)}(\hat{r}) = g_{00}(\hat{r}) + \sum_{l=1}^{\infty} \sum_{m=-l}^l [g_{lm}(\hat{r}) + B_{lm}\gamma_l(\hat{r})] Y_{lm}(\theta, \phi), \quad (34)$$

where B_{lm} are constants to be determined from the matching with the boundary layer.

Similarly, for $k = 3$ the solution can be written as

$$\psi_3^{(int)}(\hat{r}) = h_{00}(\hat{r}) + \sum_{l=1}^{\infty} \sum_{m=-l}^l [h_{lm}(\hat{r}) + C_{lm}\gamma_l(\hat{r})] Y_{lm}(\theta, \phi), \quad (35)$$

where h_{lm} are particular solutions and C_{lm} are constants, again to be determined from the matching with the boundary layer.

Because the differential operator \mathcal{D}_l and the boundary conditions at the origin are the same for the reduced radial problem of every order, we have thus obtained the general structure of the solution for the internal region (see Appendix A.1).

4.2. External Region

Here we first present the general solution and then proceed to set up the asymptotic series from equation (23). The solution to equation (18) describing the external region, i.e., in the Laplace domain, can be expressed as the sum of a particular solution $[-\epsilon T(\hat{r})]$ and of the solutions to the radial part of the Laplacian operator consistent with the boundary condition from equation (21),

$$\psi^{(ext)}(\hat{r}) = \alpha - \frac{\lambda}{\hat{r}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{\beta_{lm}}{\hat{r}^{l+1}} Y_{lm}(\theta, \phi) - \epsilon T(\hat{r}). \quad (36)$$

Here we note that the tidal potential contributes only with spherical harmonics of order $l = 0, 2$ with even values of m ,

$$T_{00}(\hat{r}) = -3\sqrt{\pi}(\nu - 1)\hat{r}^2, \quad (37)$$

$$T_{20}(\hat{r}) = 3\sqrt{\frac{\pi}{5}}(2 + \nu)\hat{r}^2, \quad (38)$$

$$T_{22}(\hat{r}) = -3\sqrt{\frac{3\pi}{5}}\nu\hat{r}^2. \quad (39)$$

At this point we can proceed to set up the asymptotic series, by expanding the constant coefficients α, λ , and β_{lm} with respect to ϵ ,

$$\alpha = aH_0 = \alpha_0 + \alpha_1\epsilon + \frac{1}{2!}\alpha_2\epsilon^2 + \dots, \quad (40)$$

$$\lambda = \lambda_0 + \lambda_1\epsilon + \frac{1}{2!}\lambda_2\epsilon^2 + \dots, \quad (41)$$

$$\beta_{lm} = a_{lm}\epsilon + \frac{1}{2!}b_{lm}\epsilon^2 + \frac{1}{3!}c_{lm}\epsilon^3 + \dots \quad (42)$$

The last expansion starts with a first-order term because the density distribution of the unperturbed problem is spherically symmetric. For convenience, we give the explicit expression of the external solution up to third order,

$$\begin{aligned} \psi^{(ext)}(\hat{r}) = & \alpha_0 - \frac{\lambda_0}{\hat{r}} + \left\{ \alpha_1 - \frac{\lambda_1}{\hat{r}} - \frac{T_{00}(\hat{r})}{2\sqrt{\pi}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \left[\frac{a_{lm}}{\hat{r}^{l+1}} \right. \right. \\ & \left. \left. + T_{lm}(\hat{r}) \right] Y_{lm}(\theta, \phi) \right\} \epsilon + \frac{1}{2!} \left[\alpha_2 - \frac{\lambda_2}{\hat{r}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{b_{lm}}{\hat{r}^{l+1}} \right. \\ & \left. \times Y_{lm}(\theta, \phi) \right] \epsilon^2 + \frac{1}{3!} \left[\alpha_3 - \frac{\lambda_3}{\hat{r}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{c_{lm}}{\hat{r}^{l+1}} Y_{lm}(\theta, \phi) \right] \epsilon^3. \end{aligned} \quad (43)$$

4.3. Boundary Layer

The boundary layer is the region where the function ψ becomes vanishingly small. Since the unperturbed gravitational field at the truncation radius is finite, $\psi_0'(\hat{r}_{tr}) \neq 0$, for any value of Ψ , based on a Taylor expansion of ψ_0 about $\hat{r} = \hat{r}_{tr}$, we may argue that the region in which the series from equation (22) breaks down can be defined by $\hat{r}_{tr} - \hat{r} = O(\epsilon)$. In this boundary layer we thus introduce a suitable change of variables,

$$\eta = \frac{\hat{r}_{tr} - \hat{r}}{\epsilon}, \quad (44)$$

take the ordering $\psi^{(lay)} = O(\epsilon)$, and thus rescale the solution by introducing the function $\tau \equiv \psi^{(lay)}/\epsilon$. For positive values of τ the Poisson equation (17) thus becomes

$$\begin{aligned} \frac{\partial^2 \tau}{\partial \eta^2} - \frac{2\epsilon}{\hat{r}_{tr} - \epsilon\eta} \frac{\partial \tau}{\partial \eta} + \frac{\epsilon^2}{(\hat{r}_{tr} - \epsilon\eta)^2} \Lambda^2 \tau \\ = -\frac{9}{\hat{\rho}(\Psi)} \epsilon \hat{\rho}(\epsilon\tau) - 9\epsilon^2(1 - \nu), \end{aligned} \quad (45)$$

where Λ^2 is the angular part of the Laplacian in spherical coordinates. For negative values of τ we can write a similar equation, corresponding to equation (18), which is obtained from equation (45) by dropping the term proportional to $\hat{\rho}(\epsilon\tau)$.

With the help of the asymptotic expansion for small arguments of the incomplete gamma function (e.g., see Bender & Orszag 1999, eq. [6.2.5]), we find

$$\hat{\rho}(\epsilon\tau) \sim \frac{2}{5}\tau^{5/2}\epsilon^{5/2} + \frac{4}{35}\tau^{7/2}\epsilon^{7/2} + \dots, \quad (46)$$

so that, within the boundary layer, the contribution of $\hat{\rho}(\epsilon\tau)$ (which is the one that distinguishes the Poisson from the Laplace regime) becomes significant only beyond the tidal term, as a correction $O(\epsilon^{7/2})$.

Therefore, up to $O(\epsilon^2)$ we can write

$$\tau = \tau_0 + \tau_1\epsilon + \frac{1}{2!}\tau_2\epsilon^2. \quad (47)$$

To this order, which is required for a full solution up to $k = 2$ of the global problem (see eqs. [22] and [23]), by equating in

equation (45) the first powers of ϵ separately, we obtain the relevant equations for the first three terms,

$$\frac{\partial^2 \tau_0}{\partial \eta^2} = 0, \quad (48)$$

$$\frac{\partial^2 \tau_1}{\partial \eta^2} = \frac{2}{\hat{r}_{\text{tr}}} \frac{\partial \tau_0}{\partial \eta}, \quad (49)$$

$$\frac{\partial^2 \tau_2}{\partial \eta^2} = \frac{4}{\hat{r}_{\text{tr}}} \left(\frac{\partial \tau_1}{\partial \eta} + \frac{\eta}{\hat{r}_{\text{tr}}} \frac{\partial \tau_0}{\partial \eta} \right) - \frac{2}{\hat{r}_{\text{tr}}^2} \Lambda^2 \tau_0 - 18(1 - \nu). \quad (50)$$

The equations are easily integrated in the variable η , to obtain the solutions

$$\tau_0 = F_0(\theta, \phi)\eta + G_0(\theta, \phi), \quad (51)$$

$$\tau_1 = \frac{F_0(\theta, \phi)}{\hat{r}_{\text{tr}}} \eta^2 + F_1(\theta, \phi)\eta + G_1(\theta, \phi), \quad (52)$$

$$\begin{aligned} \tau_2 = & \frac{2F_0(\theta, \phi)}{\hat{r}_{\text{tr}}^2} \eta^3 - \frac{1}{3\hat{r}_{\text{tr}}^2} \Lambda^2 F_0(\theta, \phi)\eta^3 + \frac{2F_1(\theta, \phi)}{\hat{r}_{\text{tr}}} \eta^2 \\ & - 9(1 - \nu)\eta^2 - \frac{1}{\hat{r}_{\text{tr}}^2} \Lambda^2 G_0(\theta, \phi)\eta^2 + F_2(\theta, \phi)\eta + G_2(\theta, \phi). \end{aligned} \quad (53)$$

The six free angular functions that appear in the formal solutions will be determined by the matching procedure.

4.4. Asymptotic Matching to 2 Orders

In order to obtain the solution, we must perform separately the relevant matching for the pairs $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ and $(\psi^{(\text{lay})}, \psi^{(\text{ext})})$. We follow the Van Dyke matching principle, which requires that we compare the second-order expansion of the internal and external solutions with the third-order expansion of the boundary layer solution. The full procedure is described in Appendix A.3.

To first order (i.e., up to $k = 1$ in series from eqs. [22] and [23]), from the matching of the pair $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ we find the free angular functions of equations (51) and (52),

$$F_0(\theta, \phi) = -\psi_0^{(\text{int})'}(\hat{r}_{\text{tr}}), \quad (54)$$

$$G_0(\theta, \phi) = \psi_1^{(\text{int})}(\hat{r}_{\text{tr}}, \theta, \phi), \quad (55)$$

$$F_1(\theta, \phi) = -\frac{\partial \psi_1^{(\text{int})}}{\partial \hat{r}}(\hat{r}_{\text{tr}}, \theta, \phi), \quad (56)$$

$$G_1(\theta, \phi) = \frac{1}{2} \psi_2^{(\text{int})}(\hat{r}_{\text{tr}}, \theta, \phi). \quad (57)$$

From the matching of the pair $(\psi^{(\text{ext})}, \psi^{(\text{lay})})$ we connect $\psi^{(\text{ext})}$ to the same angular functions, thus proving that the matching to first order is equivalent to imposing continuity of the solution up to second order and of the first radial derivative up to first order. This allows us to determine the free constants that are present in the first two terms of equation (43) and in equation (32),

$$\alpha_0 = \frac{\lambda_0}{\hat{r}_{\text{tr}}}, \quad (58)$$

$$\lambda_0 = \hat{r}_{\text{tr}}^2 \psi_0^{(\text{int})'}(\hat{r}_{\text{tr}}), \quad (59)$$

$$\alpha_1 = f_{00}(\hat{r}_{\text{tr}}) + \hat{r}_{\text{tr}} f_{00}'(\hat{r}_{\text{tr}}) + \frac{3T_{00}(\hat{r}_{\text{tr}})}{2\sqrt{\pi}}, \quad (60)$$

$$\lambda_1 = \hat{r}_{\text{tr}}^2 f_{00}'(\hat{r}_{\text{tr}}) + \frac{\hat{r}_{\text{tr}} T_{00}(\hat{r}_{\text{tr}})}{\sqrt{\pi}}, \quad (61)$$

$$A_{2m} = -\frac{5T_{2m}(\hat{r}_{\text{tr}})}{\hat{r}_{\text{tr}} \gamma_2'(\hat{r}_{\text{tr}}) + 3\gamma_2(\hat{r}_{\text{tr}})}, \quad (62)$$

$$a_{2m} = -\hat{r}_{\text{tr}}^3 [A_{2m} \gamma_2(\hat{r}_{\text{tr}}) + T_{2m}(\hat{r}_{\text{tr}})]. \quad (63)$$

Note that $A_{lm} = a_{lm} = 0$ if $l \neq 2$, for every value of m , and that the constants for $l = 2$ are nonvanishing only for $m = 0, 2$. The constants that identify the solution are thus expressed in terms of the values of the unperturbed field $\psi_0^{(\text{int})'}$, of the ‘‘driving’’ tidal potential T_{lm} , and of the solutions f_{00} and γ_2 (see eqs. [30] and [31]) taken at $\hat{r} = \hat{r}_{\text{tr}}$.

The boundary surface of the first-order model is defined implicitly by $\psi_0^{(\text{ext})}(\hat{r}) + \psi_1^{(\text{ext})}(\hat{r}, \theta, \phi)\epsilon = 0$, i.e., the spherical shape of the King model is modified by monopole and quadrupole contributions, which are even with respect to toroidal and poloidal angles and characterized by reflection symmetry with respect to the three natural coordinates planes. As might have been expected from the physical model, the spherical shape is thus modified only by spherical harmonics (l, m) for which the tidal potential has nonvanishing coefficients. Mathematically, this is nontrivial, because the first-order equation in the internal region equation (26) is nonhomogeneous only for $l = 0$; the quadrupole contribution to the internal solution is formally ‘‘hidden’’ by the use of the function ψ (which includes the tidal potential) and is unveiled by the matching which demonstrates that A_{2m} with $m = 0, 2$ are nonvanishing.

The first-order solution can be inserted into the right-hand side of equation (33) to generate nonhomogeneous equations (and thus particular solutions) only for $l = 0, 2, 4$ and corresponding positive and even values of m (see Appendix A.2). We can thus proceed to construct the second-order solution in the same way described above for the first-order solution. From the matching of the pair $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ we determine the missing angular functions,

$$F_2(\theta, \phi) = -\frac{\partial \psi_2^{(\text{int})}}{\partial \hat{r}}(\hat{r}_{\text{tr}}, \theta, \phi), \quad (64)$$

$$G_2(\theta, \phi) = \frac{1}{3} \psi_3^{(\text{int})}(\hat{r}_{\text{tr}}, \theta, \phi), \quad (65)$$

which are then connected to the properties of $\psi^{(\text{ext})}$ by the matching of the pair $(\psi^{(\text{ext})}, \psi^{(\text{lay})})$. This is equivalent to imposing continuity of the solution up to third order and of the first radial derivative up to second order and leads to the determination of the free constants that appear in the third term of equation (43) and in equation (34),

$$\alpha_2 = g_{00}(\hat{r}_{\text{tr}}) + \hat{r}_{\text{tr}} g_{00}'(\hat{r}_{\text{tr}}), \quad (66)$$

$$\lambda_2 = \hat{r}_{\text{tr}}^2 g_{00}'(\hat{r}_{\text{tr}}), \quad (67)$$

$$B_{2m} = -\frac{\hat{r}_{\text{tr}} g_{2m}'(\hat{r}_{\text{tr}}) + 3g_{2m}(\hat{r}_{\text{tr}})}{\hat{r}_{\text{tr}} \gamma_2'(\hat{r}_{\text{tr}}) + 3\gamma_2(\hat{r}_{\text{tr}})}, \quad (68)$$

$$b_{2m} = -\hat{r}_{\text{tr}}^3 [g_{2m}(\hat{r}_{\text{tr}}) + B_{2m} \gamma_2(\hat{r}_{\text{tr}})], \quad (69)$$

$$B_{4m} = -\frac{\hat{r}_{\text{tr}} g_{4m}'(\hat{r}_{\text{tr}}) + 5g_{4m}(\hat{r}_{\text{tr}})}{\hat{r}_{\text{tr}} \gamma_4'(\hat{r}_{\text{tr}}) + 5\gamma_4(\hat{r}_{\text{tr}})}, \quad (70)$$

$$b_{4m} = -\hat{r}_{\text{tr}}^5 [g_{4m}(\hat{r}_{\text{tr}}) + B_{4m} \gamma_4(\hat{r}_{\text{tr}})]. \quad (71)$$

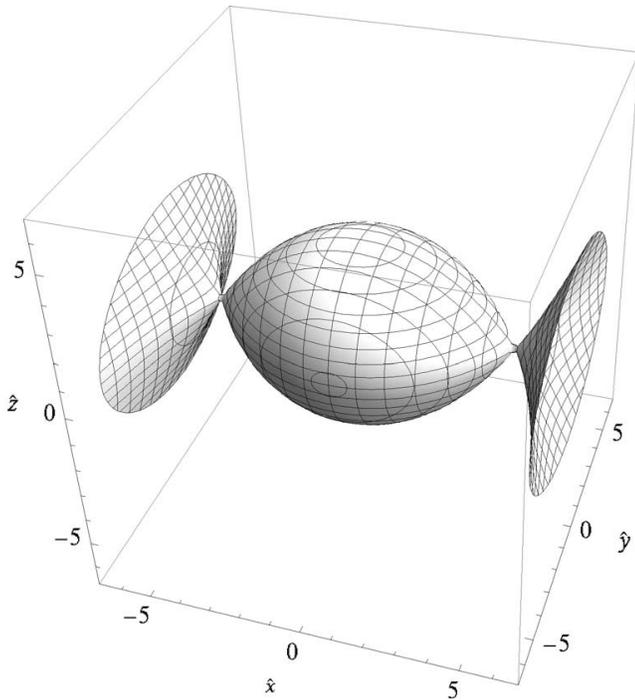


FIG. 1.— Critical Hill surface (in dimensionless variables) for a second-order model with $\Psi = 2$ and $\epsilon = 7.043 \times 10^{-4}$ (corresponding to $\delta_{\text{cr}}^{(2)} = 0.671$); the galactic potential is Keplerian ($\nu = 3$).

Here, $B_{lm} = b_{lm} = 0$ if $l \neq 2, 4$ for every value of m ; the only nonvanishing constants with $l = 2, 4$ are those with even m .

Therefore, the second-order solution has nonvanishing contributions only for $l = 0, 2, 4$, i.e., for those harmonics for which the particular solution to equation (27) is nontrivial. *By induction*, it can be proved (see Appendix A.4) that the k th-order solution is characterized by $l = 0, 2, \dots, 2k$ harmonics with corresponding positive and even values of m . In reality, the discussion of the matching to higher orders ($k > 3$) would require a redefinition of the boundary layer, because the density contribution on the right-hand side of equation (45) (for positive values of τ) comes into play. The asymptotic matching procedure carries through also in this more complex case, but for simplicity, is omitted here.

We should also keep in mind that in an asymptotic analysis the inclusion of higher order terms does not necessarily lead to better accuracy in the solution; the optimal truncation in the asymptotic series depends on the value of the expansion parameter (in this case, on the value of ϵ) and has to be judged empirically.

In conclusion, starting from a given value of the King concentration parameter Ψ and from a given strength of the tidal field ϵ , the uniform triaxial solution is constructed by numerically integrating equations (24), (30), (31), and (33) and by applying the constants derived in this subsection to the asymptotic series expansion from equations (22)–(23). The numerical integrations can be performed efficiently by means of standard Runge-Kutta routines. The boundary surface of the model is thus defined by $\psi_0^{(\text{ext})}(\hat{r}) + \psi_1^{(\text{ext})}(\hat{r}, \theta, \phi)\epsilon + \psi_2^{(\text{ext})}(\hat{r}, \theta, \phi)\epsilon^2/2 = 0$, while the internal density distribution is given by $\rho = \rho(\psi_0^{(\text{int})}(\hat{r}) + \psi_1^{(\text{int})}(\hat{r}, \theta, \phi)\epsilon + \psi_2^{(\text{int})}(\hat{r}, \theta, \phi)\epsilon^2/2)$, with the function ρ defined by equation (11). Any other “observable” quantity can be reconstructed by suitable integration in the phase space of the distribution function $f_K(H)$ defined by equation (7), with H defined by equation (5), and $\Phi_T + \Phi_C = H_0 - [\psi_0^{(\text{int})}(\hat{r}) + \psi_1^{(\text{int})}(\hat{r}, \theta, \phi)\epsilon + \psi_2^{(\text{int})}(\hat{r}, \theta, \phi)\epsilon^2/2]/a$. In Figures 1 and 2 we illustrate the main characteristics of one triaxial model constructed with the method described in this section.

5. ALTERNATIVE METHODS OF SOLUTION

5.1. The Method of Strained Coordinates

The mathematical problem described in § 3 can also be solved by the method of *strained coordinates*, an alternative method usually applied to nonlinear hyperbolic differential equations (e.g., see Van Dyke 1975, p. 106) and considered by Smith (1976) in the solution of the singular free boundary perturbation problem that arises in the study of rotating polytropes.

Starting from a series representation of the form from equations (22) and (23) for the solution defined in the Poisson and Laplace domains, respectively, a transformation is considered from spherical coordinates (\hat{r}, θ, ϕ) to “strained coordinates” (s, p, q) ,

$$\begin{aligned} \hat{r} &= s + \epsilon \hat{r}_1(s, p, q) + \frac{1}{2} \epsilon^2 \hat{r}_2(s, p, q) + \dots, \\ \theta &= p, \quad \phi = q, \end{aligned} \quad (72)$$

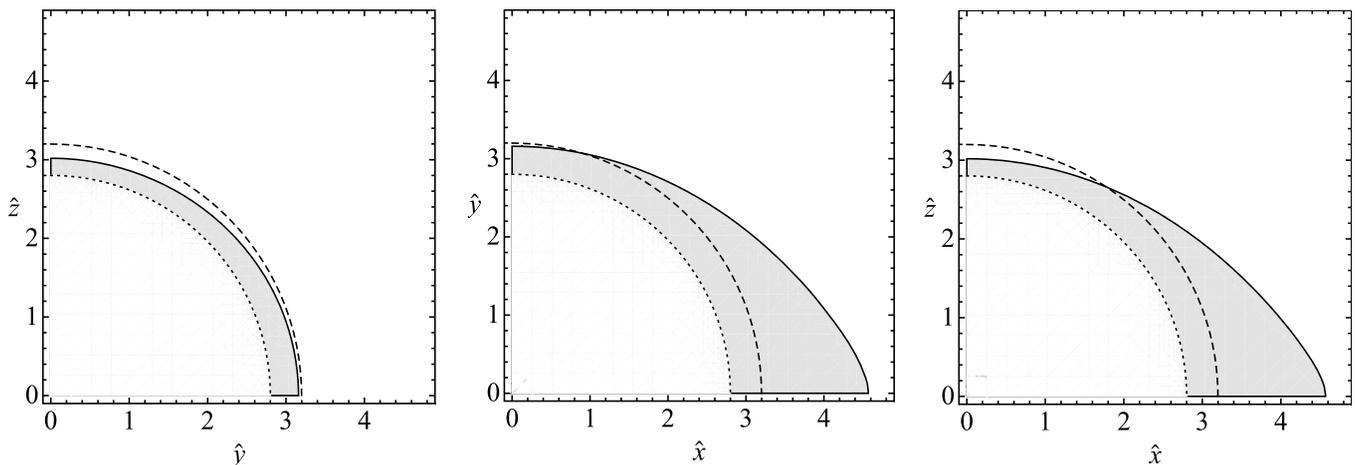


FIG. 2.— Sections in the three coordinate planes for a second-order model with $\Psi = 2$ and $\epsilon = 7.000 \times 10^{-4}$, characterized by high tidal distortion ($\delta = 0.669 \approx \delta_{\text{cr}}^{(2)}$, see Fig. 1) illustrating the boundary surface of the triaxial model (solid line), of the internal region (dotted line), and of the corresponding spherical King model (dashed line); the shaded area represents the inner region of the boundary layer. Note the compression and the elongation with respect to the unperturbed configuration in the \hat{z} - and \hat{x} -directions, respectively. The galactic potential is Keplerian ($\nu = 3$).

where $\hat{r}_k(s, p, q)$ are initially unspecified straining functions. We note that the zeroth-order problem is defined by the same equation (24) with the same boundary conditions but with the variable \hat{r} replaced by s . The unperturbed spherical boundary in the strained space is defined by $s = s_0$, where $\psi_0^{(\text{int})}(s_0) = 0$. To each order, the effective boundary of the perturbed configuration remains described by the surface $s = s_0$, while in physical coordinates the truncation radius actually changes as a result of the straining functions \hat{r}_k that are determined progressively.

The Laplacian expressed in the new coordinates, $\hat{\nabla}^2$, can be written as an asymptotic series, $\hat{\nabla}^2 = L_0 + \epsilon L_1 + 1/2\epsilon^2 L_2 + \dots$, where L_k are linear second-order operators³ in which $\hat{r}_j(s, p, q)$ (with $j = 1, \dots, k$) and their derivatives appear. For convenience, we record the zeroth- and first-order operators,

$$L_0 \equiv \frac{d^2}{ds^2} + \frac{2}{s} \frac{d}{ds}, \quad (73)$$

$$L_1 \equiv - \left(2 \frac{\partial \hat{r}_1}{\partial s} \right) \frac{d^2}{ds^2} - \left(\frac{\partial^2 \hat{r}_1}{\partial s^2} + \frac{2}{s} \frac{\partial \hat{r}_1}{\partial s} + \frac{1}{s^2} \Lambda^2 \hat{r}_1 + \frac{2}{s^2} \hat{r}_1 \right) \frac{d}{ds}, \quad (74)$$

where Λ^2 is the standard angular part of the Laplacian, written with angular coordinates (p, q) . The general k th-order operator can be decomposed as $L_k = L_1 + F_k$, where F_k is, in turn, a second-order operator in which $\hat{r}_j(s, p, q)$ (with $j = 1, \dots, k-1$) appear and L_1 is defined as in equation (74) but with $\hat{r}_k(s, p, q)$ instead of $\hat{r}_1(s, p, q)$; these operators appear in the relevant equation for $\psi_k^{(\text{int})}$,

$$[L_0 + R_1(\psi; \epsilon)]\psi_k^{(\text{int})} = L_k\psi_0^{(\text{int})}, \quad (75)$$

which corresponds to the general k th-order equation of the previous method.

Following a set of constraints that guarantee the regularity of the series from equation (22) in the strained space, the equations that uniquely identify the straining functions to any desired order can be found and solved numerically; structurally, they somewhat correspond to equations (31) and (33) in § 4.1. Therefore, the internal and external solutions can be worked out and patched by requiring continuity of the solution and of the first derivative with respect to the variable s at the boundary surface defined by $s = s_0$, in general qualitative analogy with the method described in § 4. This method is formally more elegant than the method of matched asymptotic expansions but requires a more significant numerical effort, because even though the number of equations to be solved at each order is the same, the operator that plays here a central role in the equations for the straining functions, $L_1\psi_0^{(\text{int})}$ [interpreted as an operator acting on $\hat{r}_k(s, p, q)$], is more complex than \mathcal{D}_l (defined in § 4.1).

5.2. Iteration

This technique follows the approach taken by Prendergast & Tomer (1970) and by Wilson (1975) for the construction of self-consistent dynamical models of differentially rotating elliptical galaxies and, later, by Longaretti & Lagoute (1996) for their

extension of King models to the rotating case. In terms of the function

$$u(\hat{\mathbf{r}}) \equiv a[H_0 - \Phi_C(\hat{\mathbf{r}})] = \psi(\hat{\mathbf{r}}) + \epsilon T(\hat{\mathbf{r}}), \quad (76)$$

inside the cluster the Poisson equation can be written as

$$\hat{\nabla}^2 u = -\frac{9}{\hat{\rho}(\Psi)} \hat{\rho}(u - \epsilon T), \quad (77)$$

while outside the cluster the Laplace equation is simply

$$\hat{\nabla}^2 u = 0. \quad (78)$$

The boundary conditions at the origin are $u(\mathbf{0}) = \Psi$ and $\hat{\nabla}u(\mathbf{0}) = \mathbf{0}$, because the tidal potential $T(\hat{\mathbf{r}})$ is a homogeneous function; the condition at large radii is $u \rightarrow aH_0$.

The basic idea is to get an improved solution $u^{(n)}$ of the Poisson equation by evaluating the ‘‘source term’’ on the right-hand side with the solution obtained in the immediately previous step,

$$\hat{\nabla}^2 u^{(n)} = -\frac{9}{\hat{\rho}(\Psi)} \hat{\rho}(u^{(n-1)} - \epsilon T). \quad (79)$$

The iteration is seeded by inserting as $u^{(0)}$, on the right-hand side of equation (79), the spherical solution of the King models. The iteration continues until convergence is reached.

In order to solve equation (79), we expand in spherical harmonics the solution and, correspondingly, the dimensionless density distribution,

$$u^{(n)}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l u_{lm}^{(n)}(\hat{r}) Y_{lm}(\theta, \phi), \quad (80)$$

$$\hat{\rho}^{(n)}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \hat{\rho}_{lm}^{(n)}(\hat{r}) Y_{lm}(\theta, \phi), \quad (81)$$

so that the reduced radial problems for the functions $u_{lm}^{(n)}(\hat{r})$ are

$$\left[\frac{d^2}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{d}{d\hat{r}} - \frac{l(l+1)}{\hat{r}^2} \right] u_{lm}^{(n)} = -\frac{9}{\hat{\rho}(\Psi)} \hat{\rho}_{lm}^{(n-1)}, \quad (82)$$

with boundary conditions $u_{00}^{(n)}(0) = \Psi$, $u_{lm}^{(n)}(0) = 0$, and $u_{00}^{(n)'}(0) = u_{lm}^{(n)'}(0) = 0$. Here, in contrast with the structure of the governing equations for $\psi_k^{(\text{int})}$ of §§ 4.1 and 4.2, the radial part of the Laplacian appears with no ‘‘shift,’’ for which the homogeneous solutions are known analytically. Thus, the full solution to equation (82) can be obtained in integral form by the standard method of *variation of the arbitrary constants*,

$$u_{00}^{(n)}(\hat{r}) = \Psi - \frac{9}{\hat{\rho}(\Psi)} \left[\int_0^{\hat{r}} \hat{r}' \hat{\rho}_{00}^{(n-1)}(\hat{r}') d\hat{r}' - \frac{1}{\hat{r}} \int_0^{\hat{r}} \hat{r}'^2 \hat{\rho}_{00}^{(n-1)}(\hat{r}') d\hat{r}' \right], \quad (83)$$

$$u_{lm}^{(n)}(\hat{r}) = \frac{9}{(2l+1)\hat{\rho}(\Psi)} \left[\hat{r}^l \int_{\hat{r}}^{\infty} \hat{r}'^{l-1} \hat{\rho}_{lm}^{(n-1)}(\hat{r}') d\hat{r}' + \frac{1}{\hat{r}^{l+1}} \int_0^{\hat{r}} \hat{r}'^{l+2} \hat{\rho}_{lm}^{(n-1)}(\hat{r}') d\hat{r}' \right]. \quad (84)$$

The complete calculation can be found in the Appendix of Prendergast & Tomer (1970). Here we only remark that this

³ Surfaces with constant s in the strained space are assumed to correspond to surfaces with constant $\psi^{(\text{int})}$ in the physical space, i.e., $\psi^{(\text{int})} = \psi^{(\text{int})}(s)$; therefore, L_k (with $k \geq 0$) is an ordinary differential operator for $\psi^{(\text{int})}$.

integral form is valid in both Poisson and Laplace domains, because it contains simultaneously the regular and the singular homogeneous solutions of the Laplacian. In the derivation, all the boundary conditions have been used; in particular, the two conditions at the origin are sufficient to obtain equation (83), while for equation (84), the one concerning the radial derivative at the origin is used together with the one that describes the behavior at large radii (i.e., $u_{lm}^{(n)} \rightarrow 0$ for $l \geq 1$). Furthermore, from the condition at large radii evaluated for the harmonic $l = 0$, i.e., $u_{00}^{(n)}/(4\pi)^{1/2} \rightarrow aH_0^{(n)}$ (here the notation reminds us that the value of H_0 is known only approximately and it changes slightly at every iteration), we find

$$aH_0^{(n)} \equiv \frac{\Psi}{\sqrt{4\pi}} - \frac{9}{\sqrt{4\pi}\hat{\rho}(\Psi)} \int_0^\infty \hat{r}' \hat{\rho}_{00}^{(n-1)}(\hat{r}') d\hat{r}', \quad (85)$$

where we should recall that beyond a certain radius $\hat{\rho}_{00}^{(n-1)}$ vanishes.

In terms of the function u , the boundary of the cluster is given implicitly by $u(\hat{r}) = \epsilon T(\hat{r})$. Therefore, the radial location at which the $\hat{\rho}_{lm}^{(n-1)}$ vanishes is determined numerically from

$$\hat{\rho}_{lm}^{(n-1)}(\hat{r}) = \int_0^{2\pi} \int_{-1}^1 \hat{\rho}[u^{(n-1)}(\hat{r}, \theta, \phi) - \epsilon T(\hat{r}, \theta, \phi)] \times Y_{lm}(\theta, \phi) d(\cos \theta) d\phi. \quad (86)$$

In practice, to perform the iteration, the definition of a grid in spherical coordinates and of a suitable algorithm, in order to perform the expansion and the resummation in spherical harmonics of u and $\hat{\rho}$, is required. The number of angular points of the grid and the maximum harmonic indices (l, m) admitted in the series from equations (80) and (81) are obviously related.

6. CONCLUSIONS

Spherical King models are physically justified models of quasi-relaxed stellar systems with a truncation radius argued to “summarize” the action of an external tidal field. Such simple models have had great success in representing the structure and dynamics of globular clusters, even though the presence of the tidal field is actually ignored. Motivated by these considerations and by the recent major progress in the observations of globular clusters, in this paper we have developed a systematic procedure to construct self-consistent nonspherical models of quasi-relaxed stellar systems, with special attention to models for which the nonspherical shape is due to the presence of external tides.

The procedure developed in this paper starts from a distribution function identified by replacing, in a reference spherical model, the single-star energy with the relevant Jacobi integral, thus guaranteeing that the collisionless Boltzmann equation is satisfied. Then the models are constructed by solving the Poisson equation, an elliptic partial differential equation with a free boundary. The procedure is very general and can lead to the construction of several families of nonspherical equilibrium models. In particular, we have obtained the following results.

1. We have constructed models of quasi-relaxed triaxial stellar systems in which the shape is due to the presence of external tides; these models reduce to the standard spherical King models when the tidal field is absent.

2. For these models we have outlined the general properties of the relevant parameter space; in a separate paper (A. L. Varri & G. Bertin, in preparation) we will provide a thorough description of this two-parameter family of models, also in terms of projected quantities, as appropriate for comparisons with the observations.

3. We have given a full, explicit solution to 2 orders in the tidal strength parameter, based on the method of matched asymptotic expansions; by comparison with studies of analogous problems in the theory of rotating polytropic stars, this method appears to be most satisfactory.

4. We have also discussed two alternative methods of solution, one of which is based on iteration seeded by the spherical solution; together with the use of dedicated N -body simulations, the ability to solve such a complex mathematical problem in different ways will allow us to test the quality of the solutions in great detail.

5. By suitable change of notation and physical reinterpretation, the procedure developed in this paper can be applied to the construction of nonspherical quasi-relaxed stellar systems flattened by rotation (see Appendix B).

6. The same procedure can also be applied to extend to the triaxial case other isotropic truncated models (such as low- n polytropes), that is, models that do not reduce to King models in the absence of external tides (see Appendix C).

We hope that this contribution, in addition to extending the class of self-consistent models of interest in stellar dynamics, will be the basis for the development of simple quantitative tools to investigate whether the observed shape of globular clusters is primarily determined by internal rotation, by external tides, or by pressure anisotropy.

APPENDIX A

DETAILS OF THE SOLUTION IN TERMS OF MATCHED ASYMPTOTIC EXPANSION

A1. THE GENERAL EQUATION

From the Taylor expansion about $\epsilon = 0$ of the right-hand side of equation (17), the structure of the equations for $\psi_k^{(\text{int})}$ (with $k \geq 2$) can be expressed as

$$\left[\hat{\nabla}^2 + R_1(\hat{r}; \Psi) \right] \psi_k^{(\text{int})} = - \left[\sum_{j=2}^k R_j(\hat{r}; \Psi) X_{k,j} \right], \quad (A1)$$

where $X_{k,j}$ denotes the terms that arise from the derivatives of $\psi^{(\text{int})}(\hat{r}; \epsilon)$ with respect to ϵ , thus expressed as products of $\psi_i^{(\text{int})}$ (with $i = 1, \dots, k-1$). For fixed k and j , the quantity $X_{k,j}$ is thus a sum of products of $\psi_i^{(\text{int})}$ with subscripts that are j -part partitions of the integer k . Each product of $\psi_i^{(\text{int})}$ is multiplied by a numerical factor defined as the ratio between $k!$ and the factorials of the integers that are parts of the associated partition (if an integer appears m times in the partition, the factor must also be divided by $m!$). In particular, for $k = 3$ we have

$$X_{3,2} = 3\psi_2^{(\text{int})}\psi_1^{(\text{int})}, \quad X_{3,3} = \left(\psi_1^{(\text{int})} \right)^3, \quad (A2)$$

because the 2-part partition of 3 is $2 + 1$ and the 3-part partition is trivially $1 + 1 + 1$; thus, the relevant equation is

$$\left[\hat{\nabla}^2 + R_1(\hat{r}; \Psi) \right] \psi_3^{(\text{int})} = -R_2(\hat{r}; \Psi) 3\psi_2^{(\text{int})} \psi_1^{(\text{int})} - R_3(\hat{r}; \Psi) \left(\psi_1^{(\text{int})} \right)^3. \quad (\text{A3})$$

Therefore, this formulation of the right-hand side of the general equation [together with the term $R_1(\hat{r}; \Psi) \psi_k^{(\text{int})}$ on the left-hand side] brings in the *Faà di Bruno formula* (Faà di Bruno 1855) for the k th-order derivative of a composite function in which the inner one is expressed as a series in the variable with respect to which the derivation is performed.

A2. THE EQUATION FOR THE SECOND-ORDER RADIAL PROBLEM

The expansion in spherical harmonics of $(\psi_1^{(\text{int})})^2$, which involves the product of two spherical harmonics with $l = 0$ or $l = 2$ (with m positive and even), can be performed by means of the so-called 3- j Wigner symbols.⁴ Equation (33) thus corresponds to the following set of six equations,

$$\mathcal{D}_0 \psi_{2,00}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{2\sqrt{\pi}} \left[\left(\psi_{1,00}^{(\text{int})} \right)^2 + \left(\psi_{1,20}^{(\text{int})} \right)^2 + \left(\psi_{1,22}^{(\text{int})} \right)^2 \right], \quad (\text{A4})$$

$$\mathcal{D}_2 \psi_{2,20}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{7} \sqrt{\frac{5}{\pi}} \left[\frac{7}{\sqrt{5}} \psi_{1,00}^{(\text{int})} \psi_{1,20}^{(\text{int})} + \left(\psi_{1,20}^{(\text{int})} \right)^2 - \left(\psi_{1,22}^{(\text{int})} \right)^2 \right], \quad (\text{A5})$$

$$\mathcal{D}_2 \psi_{2,22}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{\sqrt{\pi}} \left(\psi_{1,00}^{(\text{int})} \psi_{1,22}^{(\text{int})} - \frac{2\sqrt{5}}{7} \psi_{1,20}^{(\text{int})} \psi_{1,22}^{(\text{int})} \right), \quad (\text{A6})$$

$$\mathcal{D}_4 \psi_{2,40}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{7\sqrt{\pi}} \left[3 \left(\psi_{1,20}^{(\text{int})} \right)^2 + \frac{1}{2} \left(\psi_{1,22}^{(\text{int})} \right)^2 \right], \quad (\text{A7})$$

$$\mathcal{D}_4 \psi_{2,42}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{7} \sqrt{\frac{15}{\pi}} \psi_{1,20}^{(\text{int})} \psi_{1,22}^{(\text{int})}, \quad (\text{A8})$$

$$\mathcal{D}_4 \psi_{2,44}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{2} \sqrt{\frac{5}{7\pi}} \left(\psi_{1,22}^{(\text{int})} \right)^2. \quad (\text{A9})$$

A3. THE ASYMPTOTIC MATCHING FOR THE FIRST-ORDER SOLUTION

To derive the first-order solution, the matching between the pairs $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ and $(\psi^{(\text{ext})}, \psi^{(\text{lay})})$ requires that the internal (external) solution is expanded in a Taylor series around $\hat{r} = \hat{r}_{\text{tr}}$ up to terms of order $O((\hat{r} - \hat{r}_{\text{tr}})^2)$, expressed with scaled variables, expanded up to $O(\epsilon^2)$, and reexpressed with nonscaled variables,

$$\begin{aligned} (\psi^{(\cdot)})^{(2)}(\hat{r}, \theta, \phi) &= \psi_0^{(\cdot)}(\hat{r}_{\text{tr}}) - \psi_0^{(\cdot)\prime}(\hat{r}_{\text{tr}})(\hat{r}_{\text{tr}} - \hat{r}) + \frac{1}{2} \psi_0^{(\cdot)\prime\prime}(\hat{r}_{\text{tr}})(\hat{r}_{\text{tr}} - \hat{r})^2 \\ &+ \left[\psi_1^{(\cdot)}(\hat{r}_{\text{tr}}, \theta, \phi) - \frac{\partial \psi_1^{(\cdot)}}{\partial \hat{r}}(\hat{r}_{\text{tr}}, \theta, \phi)(\hat{r}_{\text{tr}} - \hat{r}) \right] \epsilon + \frac{1}{2} \psi_2^{(\cdot)}(\hat{r}_{\text{tr}}, \theta, \phi) \epsilon^2. \end{aligned} \quad (\text{A10})$$

Here, the closed parentheses include either “int” or “ext,” to denote the internal or external solution, while the notation $\psi^{(\cdot)(2)}$ on the left-hand side indicates that a second-order expansion in ϵ has been performed.

The boundary layer solution in the vicinity of $\eta = 0$ up to $O(\eta^2)$, expressed with nonscaled variables and expanded (formally) up to third order in ϵ , is given by

$$\left(\psi^{(\text{lay})} \right)^{(3)}(\hat{r}, \theta, \phi) = F_0(\theta, \phi)(\hat{r}_{\text{tr}} - \hat{r}) + \frac{F_0(\theta, \phi)}{\hat{r}_{\text{tr}}}(\hat{r}_{\text{tr}} - \hat{r})^2 + [G_0(\theta, \phi) + F_1(\theta, \phi)(\hat{r}_{\text{tr}} - \hat{r})] \epsilon + G_1(\theta, \phi) \epsilon^2. \quad (\text{A11})$$

By equating equal powers of ϵ and $(\hat{r}_{\text{tr}} - \hat{r})$ in equations (A10) and (A11), we find

$$\psi_0^{(\text{int})}(\hat{r}_{\text{tr}}) = 0 = \alpha_0 - \frac{\lambda_0}{\hat{r}_{\text{tr}}}, \quad (\text{A12})$$

$$-\psi_0^{(\text{int})\prime}(\hat{r}_{\text{tr}}) = F_0(\theta, \phi) = -\frac{\lambda_0}{\hat{r}_{\text{tr}}^2}, \quad (\text{A13})$$

⁴ For the definition of 3- j Wigner symbols and the expression of the harmonic expansion of the product of two spherical harmonics, see, e.g., Edmonds (1960, eqs. [3.7.3] and [4.6.5], respectively).

$$\frac{1}{2} \psi_0^{(\text{int})''}(\hat{r}_{\text{tr}}) = \frac{F_0(\theta, \phi)}{\hat{r}_{\text{tr}}} = -\frac{\lambda_0}{\hat{r}_{\text{tr}}^3}, \quad (\text{A14})$$

$$\psi_1^{(\text{int})}(\hat{r}_{\text{tr}}, \theta, \phi) = G_0(\theta, \phi) = \alpha_1 - \frac{\lambda_1}{\hat{r}_{\text{tr}}} - \frac{T_{00}(\hat{r}_{\text{tr}})}{2\sqrt{\pi}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \left[\frac{a_{lm}}{\hat{r}_{\text{tr}}^{l+1}} + T_{lm}(\hat{r}_{\text{tr}}) \right] Y_{lm}(\theta, \phi), \quad (\text{A15})$$

$$-\frac{\partial \psi_1^{(\text{int})}}{\partial \hat{r}}(\hat{r}_{\text{tr}}, \theta, \phi) = F_1(\theta, \phi) = -\frac{\lambda_1}{\hat{r}_{\text{tr}}^2} + \frac{T_{00}(\hat{r}_{\text{tr}})}{\sqrt{\pi} \hat{r}_{\text{tr}}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \left[\frac{(l+1)a_{lm}}{\hat{r}_{\text{tr}}^{l+2}} - \frac{2T_{lm}(\hat{r}_{\text{tr}})}{\hat{r}_{\text{tr}}} \right] Y_{lm}(\theta, \phi), \quad (\text{A16})$$

$$\frac{1}{2} \psi_2^{(\text{int})}(\hat{r}_{\text{tr}}, \theta, \phi) = G_1(\theta, \phi) = \frac{1}{2} \left[\alpha_2 - \frac{\lambda_2}{\hat{r}_{\text{tr}}} - \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{b_{lm}}{\hat{r}_{\text{tr}}^{l+1}} Y_{lm}(\theta, \phi) \right], \quad (\text{A17})$$

where the equalities on the left-hand side arise from the matching of the pair $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ and identify the free angular functions from equations (54)–(57), while those on the right-hand side arise from the matching of the pair $(\psi^{(\text{ext})}, \psi^{(\text{lay})})$. We also note that equation (A12) is consistent with the definition of the truncation radius and that equation (A14) is equivalent to equation (A13), because from equation (24) we have $\psi_0^{(\text{int})''}(\hat{r}_{\text{tr}}) = -(2/\hat{r}_{\text{tr}})\psi_0^{(\text{int})'}(\hat{r}_{\text{tr}})$.

The free constants from equations (58)–(61) are thus easily determined. For a given harmonic (l, m) of equations (A15) and (A16), with $l \geq 1$, the constants A_{lm} and a_{lm} are governed by the linear system with $i, j = 1, 2$,

$$M_{ij}u_j = v_i, \quad (\text{A18})$$

where the matrix \mathbf{M} is given by

$$\mathbf{M} = \begin{pmatrix} \gamma_l(\hat{r}_{\text{tr}}) & \hat{r}_{\text{tr}}^{-(l+1)} \\ -\gamma_l'(\hat{r}_{\text{tr}})\hat{r}_{\text{tr}} & (l+1)\hat{r}_{\text{tr}}^{-(l+1)} \end{pmatrix}, \quad (\text{A19})$$

and the vectors are defined as $(u_1, u_2) = (A_{lm}, a_{lm})$ and $(v_1, v_2) = -T_{lm}(\hat{r}_{\text{tr}})(1, -2)$. Such a linear system like equation (A18) is well posed, i.e., $\det \mathbf{M} \neq 0$.

To show this, we may integrate equation (31) for the regular solution, under the conditions $\gamma_l(0) = \gamma_l'(0) = 0$,

$$\gamma_l'(\hat{r})\hat{r}^2 = \int_0^{\hat{r}} [l(l+1) - R_1(\hat{r}; \Psi)\hat{r}^2] \gamma_l(\hat{r}) d\hat{r}. \quad (\text{A20})$$

In the vicinity of $\hat{r} = 0$ the quantity $R_1(\hat{r}; \Psi)\hat{r}^2$ is vanishingly small, so that, if the quantity $R_1(\hat{r}; \Psi)\hat{r}^2$ remains smaller than $l(l+1)$, then the regular solution, starting positive and monotonic, remains a positive and monotonically increasing function of \hat{r} . Indeed, we have checked that this condition occurs for $l \geq 2$, because for $\Psi \in [0.5, 10]$, the quantity $R_1(\hat{r}; \Psi)\hat{r}^2$ has a maximum value in the range [4.229, 3.326]. Under this condition the function $\mu_l(\hat{r}) \equiv \gamma_l'(\hat{r})\hat{r} + (l+1)\gamma_l(\hat{r})$ cannot change sign, so that $\det \mathbf{M} = \hat{r}_{\text{tr}}^{-(l+1)}\mu_l(\hat{r}_{\text{tr}})$ cannot vanish. This argument does not work for the case $l = 1$, in which the function $\mu_l(\hat{r})$ does change sign at a point $0 < \hat{r}_0 < \hat{r}_{\text{tr}}$, but we have checked directly that the property $\det \mathbf{M} \neq 0$ is satisfied also in this case.

Then equations (62) and (63) are easily recovered and we conclude that for the harmonics that are not “driven” by the tidal potential $T(\hat{r})$, the related a_{lm} and A_{lm} must vanish. Similarly, for a given harmonic (l, m) with $l \geq 1$, equation (64), which is obtained from the second-order matching, and equation (A17) can be cast in the form of equation (A18), with $(u_1, u_2) = (B_{lm}, b_{lm})$ and $(v_1, v_2) = (-g_{lm}(\hat{r}_{\text{tr}}), \hat{r}_{\text{tr}}g_{lm}'(\hat{r}_{\text{tr}}))$. Therefore, the argument provided above applies, and we can conclude that for those harmonics for which the particular solutions $g_{lm}(\hat{r})$ are absent (or, equivalently, eq. [33] is homogeneous), the constants B_{lm} and b_{lm} must vanish. A linear system equivalent to equation (A18) can be written for a fixed harmonic (l, m) with $l \geq 1$ of the solution of general order k and the same argument applies. Therefore, we conclude that the k th-order term of the solution contributes only to those harmonics for which the particular solutions are present.

A4. THE STRUCTURE OF k TH-ORDER TERM

Because we have noted (see argument introduced about the system from eq. [A18]) that the k th-order term of the solution has a nonvanishing contribution only in correspondence of those harmonics for which the component of equation (A1) is nontrivial, the discussion about the structure of the term reduces to the analysis of the structure of the expansion in spherical harmonics of the right-hand side of that equation. Recalling that the harmonic expansion of the product of two spherical harmonics (l_1, m_1) and (l_2, m_2) can be expressed by means of 3- j Wigner symbols (see Edmonds 1960, eq. [4.6.5]), we note that the composed harmonic (l, m) must satisfy the following *selection rules*: (i) $|l_1 - l_2| \leq l \leq l_1 + l_2$ (“triangular inequality”), (ii) $m_1 + m_2 = m$, and (iii) $l_1 + l_2 + l$ must be even. The last condition holds because in the cited expression the composed harmonic appears multiplied by the special case of the Wigner symbol with (l_1, l_2, l) as the first row and $(0, 0, 0)$ as the second row. Bearing in mind that the first-order term is characterized by harmonics with $l = 0, 2$ and corresponding positive and even values of m and that the structure of the right-hand side of equation (A1) can be interpreted by means of the partitions of the integer k , it can be proved by *induction* that the k th-order term is characterized by harmonics with $l = 0, 2, \dots, 2k$ and corresponding positive and even values of m .

APPENDIX B

EXTENSION TO THE PRESENCE OF INTERNAL ROTATION

It is well known that in the presence of a finite total angular momentum of the system, relaxation leads to solid-body rotation (e.g., see Landau & Lifchitz 1967). If we denote by ω the angular velocity of such rigid rotation and assume that it takes place around the z -axis, in the statistical mechanical argument that leads to the derivation of the Maxwell-Boltzmann distribution one finds that in the final distribution function the single-particle energy E is replaced by the quantity $E - \omega J_z$. Following this picture, we may consider the extension of King models to the case of internal rigid rotation. This extension is conceptually simpler than that addressed in the main text of this paper, because the perturbation associated with internal rotation, while breaking spherical symmetry, preserves axial symmetry. We note that the models described below differ from those studied by Kormendy & Anand (1971), which were characterized by a different, discontinuous truncation, and those by Prendergast & Tomer (1970) and by Wilson (1975), which were characterized by a different truncation prescription and by differential rotation.

The relevant physical model is that of a rigidly rotating isolated globular cluster characterized by angular velocity $\omega = \omega \hat{e}_z$, with respect to a frame of reference with the origin in the center of mass of the cluster. We then introduce a second frame of reference, corotating with the cluster, in which the position vector is given by $\mathbf{r} = (x, y, z)$. In such a rotating frame, the Lagrangian describing the motion of a star belonging to the cluster is given by

$$\mathcal{L} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\omega jx - 2\omega \dot{x}y) - \Phi_{\text{cen}}(x, y) - \Phi_C(x, y, z), \quad (\text{B1})$$

where $\Phi_{\text{cen}}(x, y) = -(x^2 + y^2)\omega^2/2$ is the centrifugal potential; the energy integral of the motion (called the Jacobi integral) is

$$H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_{\text{cen}} + \Phi_C. \quad (\text{B2})$$

As in the tidal case, the extension of King models is performed by considering the distribution function $f_K(H)$, as in equation (7), for $H \leq H_0$, and $f_K(H) = 0$ otherwise, with H_0 being the cutoff constant. The dimensionless energy is defined by

$$\psi(\mathbf{r}) = a\{H_0 - [\Phi_C(\mathbf{r}) + \Phi_{\text{cen}}(x, y)]\}, \quad (\text{B3})$$

and the boundary of the cluster, implicitly defined as $\psi(\mathbf{r}) = 0$, is an equipotential surface for the total potential $\Phi_C + \Phi_{\text{cen}}$. Its geometry, reflecting the properties of the centrifugal potential, is characterized by symmetry with respect to the z -axis and reflection symmetry with respect to the equatorial plane (x, y) . The constant- ψ family of surfaces, much like the Hill surfaces of the tidal case, is characterized by a *critical* surface which distinguishes the closed from the opened ones and in which the points on the equatorial plane are all saddle points. In these points the centrifugal force balances the self-gravity of the cluster; their distance from the origin, which we call the *break-off radius* (r_B), can be determined from the condition

$$\frac{\partial \psi}{\partial x}(r_B, 0, 0) = \frac{\partial \psi}{\partial y}(0, r_B, 0) = 0. \quad (\text{B4})$$

Following the argument that we gave in § 2.2 for the tidal radius, we find a zeroth-order approximation for the break-off radius, $r_B^{(0)} = (GM/\omega^2)^{1/3}$, where M is the total mass of the cluster. The discussion about the parameter space that characterizes these models is equivalent to the one presented in § 2.3, with

$$\chi \equiv \frac{\omega^2}{4\pi G\rho_0}, \quad (\text{B5})$$

a dimensionless parameter that measures the strength of the rotation with respect to the central density of the cluster, playing the role of ϵ . Similarly, we may define an extension parameter $\sigma = r_{\text{tr}}/r_B$ (instead of δ , see § 2.3). For every value of the dimensionless central potential well Ψ , a maximum value for the strength parameter, χ_{cr} , exists, corresponding to the critical condition in which the boundary of the cluster is given by the critical equipotential surface.

The relevant equations in the construction of these (fully) self-consistent models can be expressed in dimensionless form by means of the same rescaling of variables performed in § 3. Thus, for $\psi \geq 0$, the Poisson equation can be written as

$$\hat{\nabla}^2 \psi = -9 \left[\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)} - 2\chi \right], \quad (\text{B6})$$

where $\hat{\rho}$ is defined as in equation (11). For negative values of ψ we should refer to

$$\hat{\nabla}^2 \psi = 18\chi, \quad (\text{B7})$$

with the boundary conditions at the origin written as in equations (19)–(20) and at large radii given by $\psi + \chi C \rightarrow aH_0$, where $C \equiv -(9/2)(\hat{x}^2 + \hat{y}^2)$ is the dimensionless centrifugal potential.

The solution up to second order in terms of the matched asymptotic expansions presented in § 4 can be adapted to this case without effort. In fact, with respect to the calculation presented in the main text only two differences occur: (i) wherever the constant term

$-\hat{\nabla}^2 T = -9(1 - \nu)$ appears, it must be replaced here by $-\hat{\nabla}^2 C = 18$ (the sign is the same in the two cases, because $1 - \nu < 0$; see § 2.1), and (ii) thanks to axisymmetry, in the angular part of the Laplacian the derivative with respect to the toroidal angle ϕ can be dropped, and thus, the terms of the asymptotic series from equations (22)–(23) can be expanded by means of Legendre polynomials⁵ instead of spherical harmonics. The latter property implies that the radial part of each term of the asymptotic series is characterized by only one index, l , i.e., we can write $\psi_{k,l}^{(i)}$. We note that the differential operator that appears on the left-hand side of the relevant equations for the solution defined in the internal region is still \mathcal{D}_l , and thus, also the functions $\gamma_l(\hat{r})$ can be introduced in the same way as before. As to the equations corresponding to equations (51)–(53), the formal solutions of the equations in the boundary layer, the angular functions F_i and G_i (with $i = 0, 1, 2$) now depend only on the poloidal angle θ . About the external solution, an expression analogous to equation (36) can be used, with the particular solution given by χC instead of ϵT . The centrifugal potential contributes, as in the case of the tidal potential, only with monopole and quadrupole terms, explicitly

$$C_0(\hat{r}) = -3\sqrt{2}\hat{r}^2, \quad (\text{B8})$$

$$C_2(\hat{r}) = 3\sqrt{\frac{2}{5}}\hat{r}^2. \quad (\text{B9})$$

Finally, as a result of the matching of the pair $(\psi^{(\text{int})}, \psi^{(\text{lay})})$ up to second order, the expressions for the angular functions from equations (54)–(57) and (64)–(65) are still applicable. In addition, from the matching of the pair $(\psi^{(\text{lay})}, \psi^{(\text{ext})})$ up to second order, we find that the explicit expressions for the free constants follow equations (58)–(63) and (66)–(71), provided that we drop everywhere the index m and we replace $3T_{00}(\hat{r}_{\text{tr}})/(2\sqrt{\pi})$ with $3C_0(\hat{r}_{\text{tr}})/\sqrt{2}$ in equation (60) and $T_{00}(\hat{r}_{\text{tr}})/(\sqrt{\pi}\hat{r}_{\text{tr}})$ with $2C_0(\hat{r}_{\text{tr}})/(\sqrt{2}\hat{r}_{\text{tr}})$ in equation (61). Obviously, the particular solutions f_0 and g_l (with $l = 0, 2, 4$) are different from the ones obtained in the tidal case, because the right-hand side of the relevant equations is different. Also in this case, it can be proved *by induction* that the k th-order term has nonvanishing contributions only for $l = 0, 2, \dots, 2k$.

For completeness, we record the explicit expression of the second-order equations in the internal region,

$$\mathcal{D}_0\psi_{2,0}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{1}{\sqrt{2}} \left[\left(\psi_{1,0}^{(\text{int})} \right)^2 + \left(\psi_{1,2}^{(\text{int})} \right)^2 \right], \quad (\text{B10})$$

$$\mathcal{D}_2\psi_{2,2}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{\sqrt{10}}{7} \left[\frac{7}{\sqrt{5}} \psi_{1,0}^{(\text{int})} \psi_{1,2}^{(\text{int})} + \left(\psi_{1,2}^{(\text{int})} \right)^2 \right], \quad (\text{B11})$$

$$\mathcal{D}_4\psi_{2,4}^{(\text{int})} = -R_2(\hat{r}; \Psi) \frac{3\sqrt{2}}{7} \left(\psi_{1,2}^{(\text{int})} \right)^2. \quad (\text{B12})$$

We remark that the Legendre expansion of the product of two Legendre polynomials is straightforward, because the $3-j$ Wigner symbols of interest all belong to the special case with $(0, 0, 0)$ as the second row.

APPENDIX C

EXTENSION OF OTHER ISOTROPIC TRUNCATED MODELS

The procedure developed in §§ 3 and 4 can be applied also to extend other isotropic truncated models, different from the King models, to the case of tidal distortions. Here we briefly describe the case of low- n polytropes ($1 < n < 5$), which are particularly well suited to the purpose, because they are characterized by a very simple analytical expression for the density as a function of the potential; this class of models was also considered by Weinberg (1993). In the distribution function that defines the polytropes (e.g., see Bertin 2000), we may thus replace the single-star energy with the Jacobi integral (see the definition from eq. [5]) and consider

$$f_P(H) = A(H_0 - H)^{n-3/2}, \quad (\text{C1})$$

for $H \leq H_0$, and a vanishing distribution otherwise. Unlike the King family discussed in the main text, these models have no dimensionless parameter to measure the concentration of the stellar system, which depends only on the polytropic index n ; in fact, the spherical fully self-consistent polytropes are characterized only by two physical scales, which are associated with the cutoff constant H_0 and the normalization factor A . Below, we consider values of $n < 5$, so that the models have a finite radius. Therefore, the relevant parameter space for the tidally distorted models is represented just by the tidal strength parameter ϵ (see definition from eq. [13]), which, for a given value of the index n , has a (maximal) critical value. The definition from equation (14) for the extension parameter δ is still valid if we denote the radius of the unperturbed spherical configuration by r_{tr} . The associated density functional is given by

$$\rho(\psi) = \rho_0 \psi^n, \quad (\text{C2})$$

where the dimensionless escape energy is given by

$$\psi(\mathbf{r}) = \{H_0 - [\Phi_C(\mathbf{r}) + \Phi_T(\mathbf{r})]\} \left(\frac{c_n}{\rho_0} \right)^{1/n}, \quad (\text{C3})$$

⁵ Following Abramowitz & Stegun (1964) we use Legendre polynomials as defined in eq. (22.3.8), i.e., with the Condon-Shortley phase, and normalized with respect to eq. (22.2.10). We remark that, although they are structurally equivalent to zonal spherical harmonics, the normalization is different.

with $c_n \equiv (2\pi)^{3/2} \Gamma(n-1/2) A/n!$. The boundary of the perturbed configuration is defined by $\psi(\mathbf{r}) = 0$, following the same arguments described in the main text. Here ρ_0 can be interpreted as the central density if we set $\psi(\mathbf{0}) = 1$.

For $\psi \geq 0$, the relevant equation for the construction of the self-consistent tidally distorted models is the Poisson equation, which, in dimensionless form, is given by

$$\hat{\nabla}^2 \psi = -[\psi^n - \epsilon(1 - \nu)], \quad (\text{C4})$$

while for negative values of ψ we must refer to equation (18). Here the rescaling of variables has been performed by means of the scale length $\zeta \equiv [\rho_0^{1/n-1}/(4\pi G c_n^{1/n})]^{1/2}$. The relevant boundary conditions are given by $\psi(\mathbf{0}) = 1$ instead of equation (19), while equations (20) and (21) hold unchanged.

If the polytropic index is in the range $1 < n < 5$, the solution up to second order presented in § 4 is fully applicable, provided that we note that the problem for the zeroth-order term of the series from equation (22) is now given by the Lane-Emden equation (see, e.g., Chandrasekhar 1939),

$$\psi_0^{(\text{int})''} + \frac{2}{\hat{r}} \psi_0^{(\text{int})'} = -\left(\psi_0^{(\text{int})}\right)^n, \quad (\text{C5})$$

with $\psi_0^{(\text{int})}(0) = 1$ and $\psi_0^{(\text{int})'}(0) = 0$, where the prime denotes a derivative with respect to the argument \hat{r} ; explicitly, the truncation radius \hat{r}_{tr} is now defined by $\psi_0^{(\text{int})}(\hat{r}_{\text{tr}}) = 0$, i.e., it represents the radius of the so-called *Emden sphere*. Correspondingly, the quantities called R_j in the main text must be redefined as

$$R_j(\hat{r}; n) \equiv \left. \frac{d^j \psi^n}{d\psi^j} \right|_{\psi_0^{(\text{int})}}; \quad (\text{C6})$$

the value of j at which the quantity R_j may start to diverge depends on the index n . Obviously, in equation (45), i.e., in the Poisson equation defined in the boundary layer, $\hat{\rho}(\epsilon\tau)$ must be replaced by $(\epsilon\tau)^n$. This makes it clear that the value of the polytropic index n directly affects the order, with respect to the perturbation parameter, at which the density contribution on the right-hand side of equation (45) comes into play and therefore changes the matching procedure. If $n > 1$, the density contribution emerges only after the second order, and thus, the full procedure described in § 4 is valid. In contrast, if $n \leq 1$, the procedure described in the main text is applicable only up to first order, while the calculation of second-order terms would require a redefinition of the boundary layer (as it happens for the case discussed in the main text when terms of order $k > 3$ are desired). In closing, we note that the procedure presented in this paper can be applied also to isotropic truncated models with more complicated expressions for the density functional (e.g., the family of models f_{1n} proposed by Davoust [1977] without boundary conditions on tangential velocity, for which the density functionals are expressed in terms of the error function and of the Dawson integral), bearing in mind the last *caveat* about the possibility that the density contribution may affect at some order the boundary layer, thus requiring a reformulation of the results presented in § 4.

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